Penney Ante

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CBL Tea Talk
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  - what's the chance of HH? $\frac{1}{4}$
  - what's the chance of HT? $\frac{1}{4}$ same
- Keep flipping until we get HH, what's the expected number of flips? How about if we flip until we get HT - is it the same?

Expected waiting times

- Let t_{HH} be $\mathbb{E}$[num flips to HH], then

$$t_{HH} = \frac{1}{4}\mathbb{E}[\text{num}|HH] + \frac{1}{4}\mathbb{E}[\text{num}|HT] + \frac{1}{4}\mathbb{E}[\text{num}|TT] + \frac{1}{4}\mathbb{E}[\text{num}|TH]$$
$$= \frac{1}{4}\left(2 + (2 + t_{HH}) + (2 + t_{HH}) + \frac{1}{2}.3 + \frac{1}{2}.(3 + t_{HH})\right)$$
$$= \frac{1}{4}\left(9 + \frac{5}{2}t_{HH}\right)$$
$$\Rightarrow t_{HH} = 6$$

- Let t_{HT} be $\mathbb{E}$[num flips to HT], then

$$t_{HT} = \frac{1}{4}\mathbb{E}[\text{num}|HT] + \frac{1}{4}\mathbb{E}[\text{num}|TT] + \frac{1}{4}\mathbb{E}[\text{num}|TH] + \frac{1}{4}\mathbb{E}[\text{num}|HH]$$
$$= \frac{1}{4}\Big(2 + (2 + t_{HT}) + 2.\,[2 + t_{H\to HT}]\Big),\ t_{H\to HT} = \frac{1}{2}.1 + \frac{1}{2}.(1 + t_{H\to HT})$$
$$\Rightarrow t_{HT} = 4$$

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- Penney's game (nontransitive): for sequences of ≥ 3 tosses, the second player can always do better

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- Suppose the first player chooses $A - B - C$ $\quad A, B, C \in \{\text{H,T}\}$
- You should choose $\bar{B} - A - B$
- Intuition: Unless the first player gets his/her sequence in the first 3 tosses, often if $A - B$ occurs, you will already have won
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First player choice	Your choice	Prob(You win)
HHH	THH	$\frac{7}{8}$
HHT	THH	$\frac{3}{4}$
HTH	HHT	$\frac{2}{3}$
HTT	HHT	$\frac{2}{3}$
THH	TTH	$\frac{2}{3}$
THT	TTH	$\frac{2}{3}$
TTH	HTT	$\frac{3}{4}$
TTT	HTT	$\frac{7}{8}$

Conclusion

- You now have special knowledge, use it responsibly
- On the other hand, if you find yourself in a situation where you're not sure what's going on...

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- On the other hand, if you find yourself in a situation where you're not sure what's going on… find a reason not to play

Thank you

References

- Penney's game. Wikipedia. https://en.wikipedia.org/wiki/Penney%27s_game
- YouTube scam school. https://www.youtube.com/watch?v=IMsa-qBlPIE
- E. Berlekamp, J. Conway and R. Guy. Winning ways for your mathematical plays, 2nd Ed, Vol 4, AK Peters, p.885, 2004.
- J. Csirik. Optimal strategy for the first player in the Penney ante game. Combinatorics, Probability and Computing, Volume 1, Issue 4, pp.311-321, 1992.
- S. Humble and Y. Nishiyama. Humble-Nishiyama randomness game - a new variation on Penney's coin game. IMA Mathematics Today. Vol 46, No. 4, pp.194-195, August 2010.
- R. Nickerson. Penney Ante: counterintuitive probabilities in coin tossing. The UMAP Journal. Vol 28, No. 4, p.503, 2007
- Y. Nishiyama. Pattern matching probabilities and paradoxes as a new variation on Penney's coin game. Intl Jrnl of Pure and Applied Math. Vol 59, No. 3, pp.357-366, 2010
- W. Penney. Problem 95. Penney-Ante. Journal of Recreational Mathematics, 2, p.241, 1969.

More fun

- John Conway developed a beautiful algorithm that quickly computes the odds of one sequence coming before another
  - For sequences A and B (any length), odds of B coming first are $\frac{|AA|-|AB|}{|BB|-|BA|}$ using a clever overlap metric, ask if interested [Nish10]
- Is it possible to have two sequences where the first has expected waiting time $<$ the second yet on average, the second will occur before the first?

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  - Let $S_1 =$'HTHH', $S_2 =$'THTH' then $t_1 = 18$, $t_2 = 20$ but prob(THTH before HTHH)$= \frac{9}{14} \approx 64\%$
- What happens if instead of using one coin, each player tosses their own coin?

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- What happens if instead of using one coin, each player tosses their own coin? The sequence with shorter waiting time wins