Revisiting the Limits of MAP Inference by MWSS on Perfect Graphs

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SUMMARY

We address *exact MAP inference* for undirected graphical models, i.e. finding a global mode configuration with highest probability. We are interested in when this is *efficient*, i.e. solvable in time polynomial in the number of variables.

We focus on *binary pairwise* (Ising) models, e.g. vision, RBMs, or social networks, and explore the limits of an exciting recent method (Jebara, 2009):

- Reduce the problem to finding a *maximum weight stable set* (MWSS) in a derived weighted graph called a *nand Markov random field* (NMRF);
- This approach is efficient if the *pruned* NMRF is a *perfect graph*.

RESULTS

Only a few signed topologies were known always to admit efficient MAP inference:

- Acyclic models (via dynamic programming),
- Attractive models, i.e. all edges attractive (via graph cuts or LP relaxation)
  - ▷ Generalizes to *balanced* models (no *frustrated cycles*, see below).

These were previously shown to be solvable via a perfect NMRF. Here we go further to establish the following result, which defines the power of the approach:

Theorem (main result)

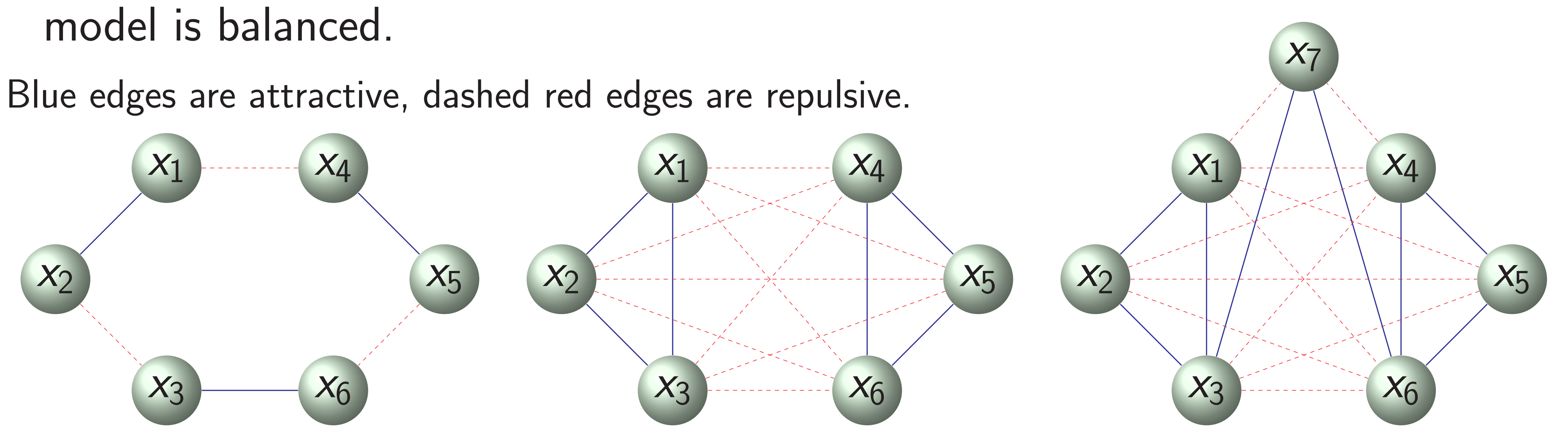
A binary pairwise model maps efficiently to a perfect pruned NMRF for any valid potentials iff each *block* of the model is balanced or *almost balanced*.

- Our approach is the only method known to solve all such models efficiently.

FRUSTRATED, BALANCED, ALMOST BALANCED

- Each edge of a model may be characterized as attractive (pulls variables toward the same value) or repulsive (pushes variables apart to different values).
- A *frustrated cycle* contains an odd number of repulsive edges. These are challenging for many methods of inference.
- A *balanced* model contains no frustrated cycle ⇔ its variables form two partitions with all intra-edges attractive and all inter-edges repulsive.
- An *almost balanced* model contains a variable s.t. if it is removed, the remaining model is balanced.

Blue edges are attractive, dashed red edges are repulsive.



frustrated cycle | balanced | almost balanced

REFERENCES

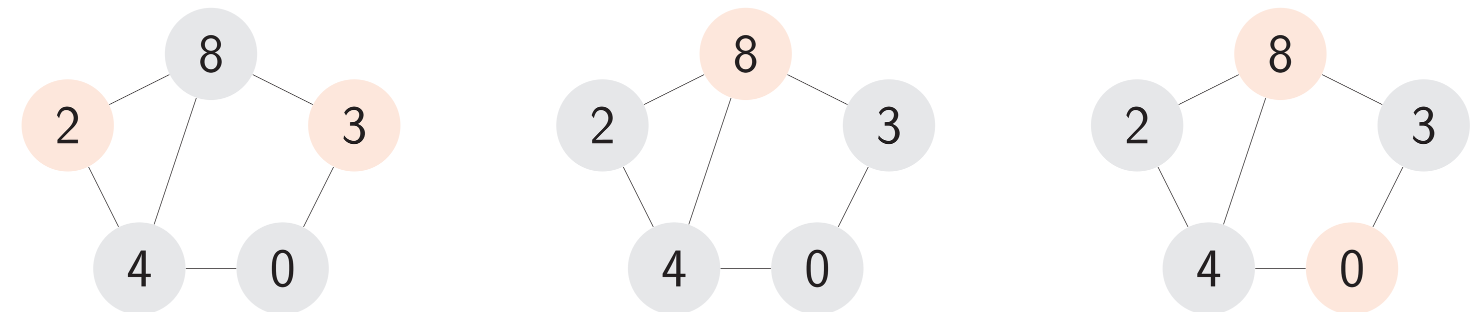
M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas. The strong perfect graph theorem. *Ann. Math*, 164:51–229, 2006.

T. Jebara. MAP estimation, message passing, and perfect graphs. In *UAI*, 2009.

A. Weller and T. Jebara. On MAP inference by MWSS on perfect graphs. In *UAI*, 2013.

STABLE SETS, MWSS IN WEIGHTED GRAPHS

A set of (weighted) nodes is *stable* if no two are adjacent.



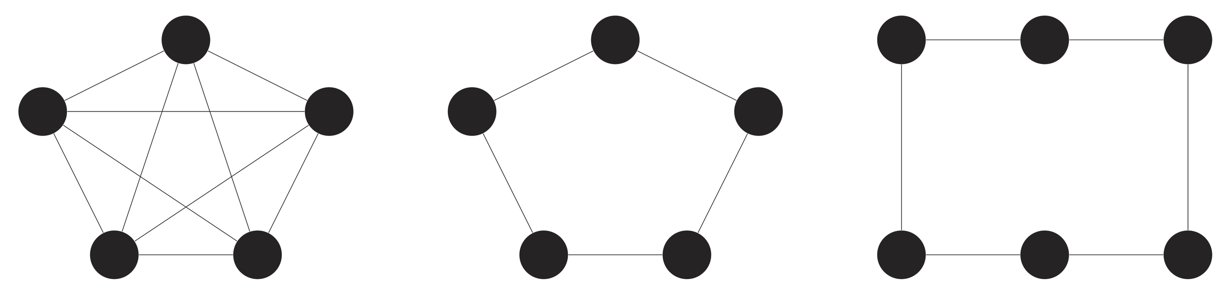
Stable set | Max Weight Stable Set (MWSS) | Maximal MWSS (MMWSS)

- Finding a MWSS is NP-hard in general, but is known to be efficient for *perfect* graphs.

PERFECT GRAPHS

Berge defined perfect graphs in 1960. The Strong Perfect Graph Theorem (Chudnovsky et al., 2006) yields an alternative definition:

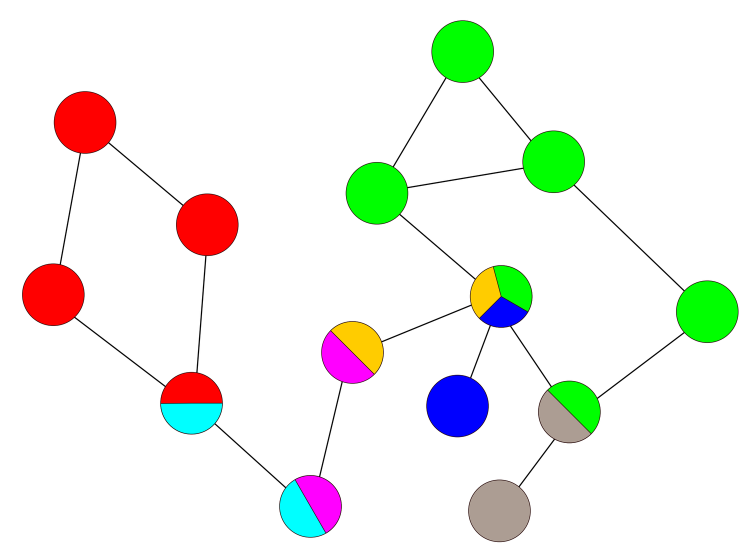
- A graph is *perfect* iff it contains no *odd hole* or *odd antihole*.
- An *odd hole* is an induced subgraph which is a (chordless) odd cycle of length ≥ 4.
- An *antihole* is the complement of a hole (each edge of antihole is present iff not present in hole).



perfect | not perfect (odd hole) | perfect (even hole)

BLOCK DECOMPOSITION

A *cut vertex* of a connected graph is a vertex s.t. removing it disconnects the graph. A *block* is a maximal subgraph that does not contain a cut vertex. A graph may be repeatedly split apart at cut vertices until what remains is the unique *block decomposition*.
In general, this could contain $\Omega(|\text{number of vertices}|)$ blocks.



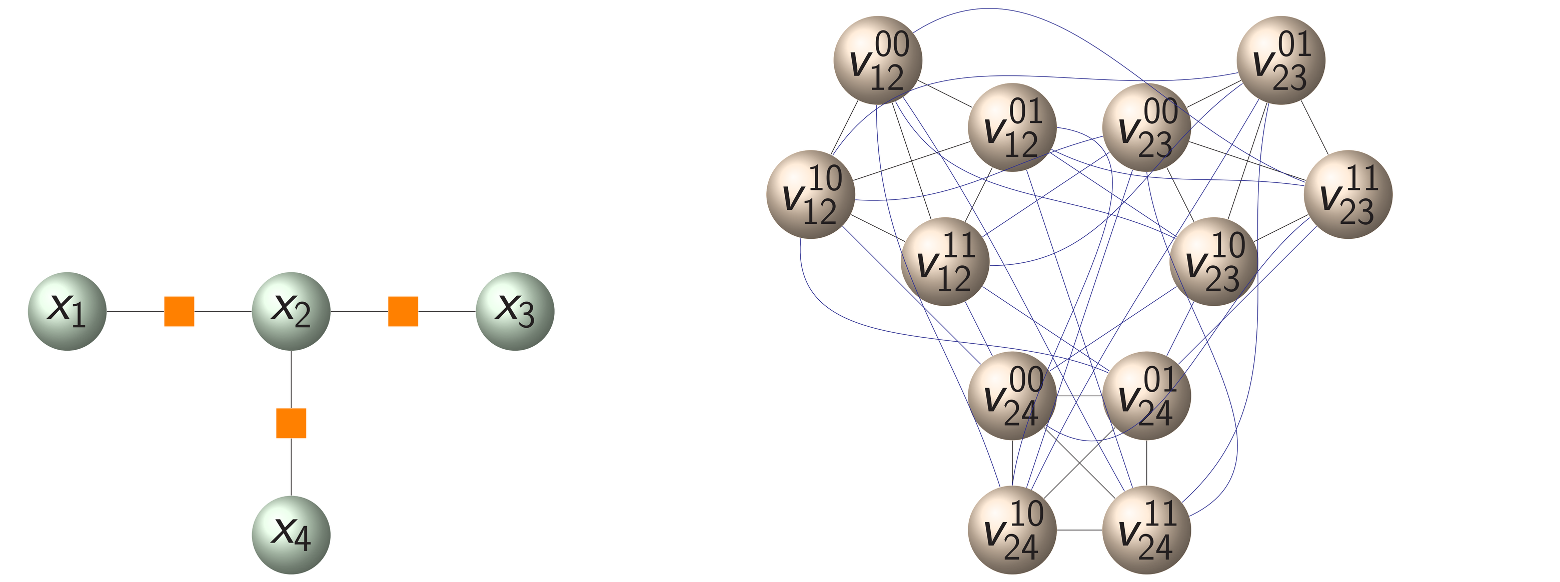
Example image from wikipedia.

Here each color indicates a block. Multi-colored vertices are cut vertices, hence belong to multiple blocks.

REDUCTION TO MWSS ON AN NMRF

Given a model with potentials $\{\psi_c\}$ over variable sets $\{c\}$, construct a *nand Markov random field* (NMRF, Jebara, 2009) N, defined as follows:

- A weighted graph $N(V_N, E_N, w)$ with vertices V_N, edges E_N and a weight function $w : V_N \to \mathbb{R}_{\geq 0}$.
- Each c of the original model maps to a clique in N. This contains one node for each possible configuration x_c, with all these nodes pairwise adjacent in N.
- Nodes in N are adjacent iff they have inconsistent settings for any variable X_i.
- Nonnegative weights of each node in N are set as $\psi_c(x_c) - \min_{x_c} \psi_c(x_c)$, hence the minimum weight is zero which facilitates *pruning*.



Original model (factor graph) | Derived NMRF

In NMRF, subscripts denote variable set c, superscripts denote configuration x_c.

Intuition: A MAP configuration has $\max_x \sum_c \psi_c(x_c) = \sum_c \max_{x_c} \psi_c(x_c)$ s.t. all the x_c settings are consistent, which is enforced by requiring a stable set.

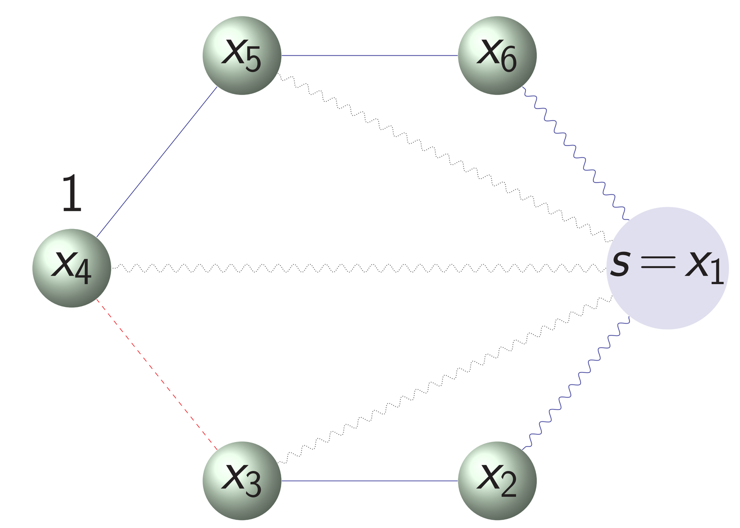
Earlier results:

- A MMWSS of the NMRF returns a MAP configuration of the original model.
- To find a MMWSS of the NMRF: zero-weight nodes may be *pruned* (removed), a MWSS found, then zero-weight nodes added back greedily.
- MAP inference is efficient if ∃ an efficiently identifiable efficient reparameterization s.t. the model maps to a perfect pruned NMRF.
- Decomposition: If each *block* of a model yields a perfect NMRF, then so too will the whole model (Weller and Jebara, 2013).

EXAMPLE APPLICATION TO A FRUSTRATED CYCLE

In the paper, we show constructively how MAP inference may be performed efficiently for any model composed of (possibly many) almost balanced blocks.

Here we illustrate the approach for one almost balanced block. Blue edges are attractive, dashed red are repulsive. Straight edges are reparameterized s.t. they lead to one node in the pruned NMRF, wiggly edges may have all 4 possible nodes. Gray edges are 'phantom edges' introduced to absorb nodes from singleton potentials. The special vertex s was chosen as x_1, removing this renders the remaining graph balanced (in fact acyclic in this example). Marks are shown next to their vertices for the two partitions in the balanced portion of the model. See paper for details.



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Download the full paper at http://www.jmlr.org/proceedings/papers/v38/weller15.pdf

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