Estimation of Financial Indices Volatility Using a Model with Time-Varying Parameters

Felipe A. Tobar
Imperial College London, UK.
f.tobar@imperial.ac.uk

Marcos E. Orchard
Universidad de Chile, Chile.
morchard@ing.uchile.cl

Danilo P. Mandic and Anthony G. Constantinides
Imperial College London, UK.
{d.mandic, a.constantineides}@imperial.ac.uk

Abstract—A class of stochastic volatility models (SVMs) with time-varying parameters is presented for online volatility estimation in nonstationary environments. This is achieved by modelling both the volatility and model parameters as states of a hidden Markov model (HMM), thus allowing for the use of particle filters to estimate the resulting posterior densities. The proposed models, based on the logarithmic SVM and the unobserved GARCH model, are evaluated for the estimation of the volatility of the NASDAQ-C and the Chilean IGPA financial indices between June 2007 and January 2010, where the late-2000s financial crisis is included. Simulations show that the proposed time-varying HMMs achieve levels of accuracy comparable to those of offline (batch) algorithms, and (ii) their parameters can be used to identify market changes.

I. INTRODUCTION

Volatility models describe the evolution of the volatility and its relationship to observed returns. In the design of such models, practitioners aim to account for most of the observed data features (stylised facts) so as to provide models able to explain future observations [1], however, there is a trade-off between the model accuracy and simplicity that hinders system identification in real-world applications.

A well-known volatility model is the generalised autoregressive conditional heteroskedasticity (GARCH) model [2], which assumes that the system volatility is driven by the observed returns in a deterministic fashion. Conversely, from a Bayesian standpoint, the observed returns can be related to their volatility via a hidden Markov model (HMM), whereby the volatility is the hidden state and the return series is the observed process; this class of models is referred to as stochastic volatility models (SVMs) [3]. Within this formulation, it is possible to account for the stylised facts associated to financial returns, such as higher order moments and volatility clustering [4], [5], by including non-linearities, non-Gaussian noise, and unobservable states. The main drawback of SVMs is model identification, since their likelihood function is defined as an intractable integral whose maximisation is not straightforward. Furthermore, even if iterative methods such as expectation-maximisation [6] are used to identify a meaningful set of parameters based on historical training data, these may not provide a reliable estimate for different datasets due to the nonstationary nature of financial markets, thus suggesting the use of time-varying parameters.

Although enhanced modelling can be achieved based on both sample statistics and time-varying parameters, the resulting mathematical models are complex and do not admit close-form solutions as in the linear and Gaussian case [7]. Particle filters (PF) provide a flexible and accurate framework for estimating the hidden process of nonlinear, non-Gaussian, and high dimensionality HMMs [8], [9]. Moreover, PF can also be used to address the problem of joint state (volatility) and parameter estimation [10], [11] by regarding the HMM state as a vector containing both the volatility and the parameters (hyperstate). A recent application of PF to find the mixing parameters of radial basis functions is given in [12].

Previous research on GARCH and SVMs includes quantification of the observed errors in option prices and prediction of risk in hypothetical scenarios [13], and a comparison between a stochastic version of the GARCH model and the logarithmic SVM model [14]. These studies state that GARCH-based models provide reliable estimates in pricing and risk estimation. However, they do not refer to the estimation of confidence intervals or how the models behave under a change of the dynamics in the observed price series.

We focus on the joint problem of volatility estimation and adaptive model identification, by proposing time-varying extensions of two stochastic volatility models, the logarithmic SVM [3] and the unobserved GARCH [15]. Our aim is to show that the proposed time-varying HMMs achieve levels of accuracy comparable to those of offline methods (which require past and future observations to estimate the volatility and the parameters), thus providing reliable online estimates based on (only) past observations. Additionally, we aim to provide physically meaningful parameter estimates explaining the observed changes in the return series at early stages, paving the way for the identification of market anomalies.

This paper is organised as follows, Section II explains the relationship between returns and volatility via the GARCH model, while Section III introduces two stochastic volatility models: the logarithmic SVM and a GARCH-based SVM referred to as the unobserved GARCH. Section IV addresses the joint model identification and state estimation problem. Section V shows simulations where the proposed time-varying models were used to estimate the volatility of two financial indices, the NASDAQ-C and the Chilean IGPA, whereas Section VI concludes this work.
II. FINANCIAL RETURNS SERIES AND THE GARCH MODEL

The design of volatility models can benefit from the empirical findings (stylised facts) of the available data. These include (i) the so-called volatility clustering, that is, the presence of alternating periods of either high or low volatility [16], and (ii) the presence of non-Gaussian, peaked and asymmetric sample distributions (histograms) of the returns, which are associated to high skewness and kurtosis. To model these stylised facts, the return $r_t$ can be modelled as a product of two processes, that is,

$$r_t = \sigma_t \epsilon_t,$$

where $\sigma_t$ is the volatility of the return and $\{\epsilon_t\}_{t \in \mathbb{N}}$ is a white noise sequence, independent of $\{\sigma_t\}_{t \in \mathbb{N}}$ and $\{r_t\}_{t \in \mathbb{N}}$. Eq. (1) can be interpreted as the observation equation of a state space model, whereby the state equation is a recursive formula for the state $\sigma_t$.

The generalised autoregressive conditional heteroskedasticity model (GARCH) [2] estimates the variance $\sigma_t^2$ using an autoregressive model driven by the observed return series $r_0, r_1, \ldots$. The estimate is denoted by $\sigma^2_{t|t-1}$, and the model given by

$$\sigma^2_{t|t-1} = \omega + \beta \sigma^2_{t-1|t-2} + \alpha u^2_t - 1$$

$$r_t = \mu + u_t,$$

where $\omega > 0$, $\alpha, \beta \geq 0$, $\mu \in \mathbb{R}$ are the model parameters, and the return $r_t$ is the observation. The innovation $u_t = \sigma_t \epsilon_t$ is a product between the white noise sequence $\{\epsilon_t\}_{t \in \mathbb{N}}$ and the current estimate of the volatility. For convenience, it is usually assumed that $\epsilon_t \sim \mathcal{N}(0, 1)$, or equivalently for the return bias,

$$r_t - \mu = u_t \sim \mathcal{N}(0, \sigma^2_{t|t-1}).$$

The GARCH model has been extensively used due to its ability to represent the aforementioned stylised facts and ease of identification based on maximum likelihood (ML), however, its deterministic nature does not allow for a full statistical description of the volatility.

III. STOCHASTIC VolATILITY MODELS

Long-term observations show clear, and apparently random, changes in volatility that are not entirely explained by historical data [17]. To account for these findings, stochastic volatility models (SVM) assume that the volatility is driven by an innovation sequence that is independent of the observations. In practice, this concept is implemented by modelling financial returns series using a hidden Markov model (HMM), whereby the volatility (or variance) is the hidden state and the observed return is the system output. Unlike deterministic models (e.g. GARCH), which only provide point estimates, within the HMM formulation standard nonlinear filtering techniques can be used to estimate the complete posterior distribution, thus allowing for the calculation of expectations and confidence intervals.

We next review the logarithmic SVM, and present a stochastic extension of the GARCH model. The identification issue in stochastic volatility models is also addressed.

A. The logarithmic stochastic volatility model (log-SV)

Empirical distributions constructed upon observed returns have shown the presence of extreme values located in the right side of the mean, which can be explained by innovation sequences distributed according to a right-skewed positive-support density, such as the log-normal distribution $\log\mathcal{N}(\cdot, \cdot)$ [18], [19]. The return variance $\sigma_t^2$ can therefore be assumed to be log-normally distributed with parameters $m$ and $\sigma$, that is,

$$\sigma_t^2 \sim \log\mathcal{N}(m, \sigma),$$

or equivalently, $V_t = \log \sigma_t^2 \sim \mathcal{N}(m, \sigma)$.

By assuming a first order, linear recurrence for $V_t$, $m$ is defined as $m = \gamma + \phi V_{t-1}$, where $\gamma$ is known as the unconditional log-variance and the coefficient $\phi \in (-1, 1)$ represents the correlation between consecutive values of $V_{1:t}$. The log-SV model [3] is thus given by

$$V_t = \gamma + \phi V_{t-1} + \eta_t$$

$$r_t = \exp\left(\frac{1}{2} V_t\right) \epsilon_t,$$

where the return $r_t$ is the observation, the log-variance $V_t$ is the state of the system, and $\{\epsilon_t\}_{t \in \mathbb{N}} \sim \mathcal{N}(0, 1)$, $\{\eta_t\}_{t \in \mathbb{N}} \sim \mathcal{N}(0, \sigma)$ are independent noise sequences.

B. The unobserved GARCH

To design a GARCH-based SVM, the volatility $\sigma_t$ can be modelled as a stochastic process driven by a Gaussian innovation process $\eta_t$, rather than the observed shocks $u_t = r_t - \mu$. The supporting idea behind the this concept stems from the belief that the process $\{\sigma_t\}_{t \in \mathbb{N}}$ is best described by a latent driving signal, instead of the observed shock. Therefore, as the observed return does not appear explicitly in the state equation, we refer to this model as the unobserved GARCH (uGARCH) [15].

The uGARCH model is given by

$$\sigma_t^2 = \omega + \beta \sigma^2_{t-1} + \alpha \epsilon^2_{t-1} \eta_{t-1}, \quad \eta_t \sim \mathcal{N}(0, 1)$$

$$r_t = \mu + \sigma_t \epsilon_t, \quad \epsilon_t \sim \mathcal{N}(0, 1)$$

where $r_t$ is the return process, $\sigma_t$ is the stochastic volatility, and the random variables $\epsilon_t$ and $\eta_t$ are independent. Following the conventional GARCH, $\mu \in \mathbb{R}$, $\omega > 0$, $\alpha \geq 0$, and $\beta \geq 0$ so that $\sigma_t^2 > 0$ for any value of $\eta_t$.

Remark 1. Observe that under the uGARCH, the volatility $\sigma_t$ is not entirely determined by $r_t$, therefore, the notation $\sigma_t$ has been considered instead of $\sigma_{t|t-1}$.

The uGARCH inherits the standard GARCH’s ability to explain the stylised fact of the return series, while at the same time allowing for a statistical description of the volatility. In particular, the uGARCH volatility is driven by independent positive shocks distributed according to $\chi^2$, thus having positive skewness and excess kurtosis.
C. Parametric identification of stochastic volatility models

By denoting the SVM parameters by \( \theta = [\omega, \mu, \alpha, \beta] \) for uGARCH and \( \theta = [\gamma, \phi] \) for log-SV, the likelihood function of the SVM, \( L(\theta|r_{1:t}) \), is given by

\[
L(\theta|r_{1:t}) = p(r_{1:t}\mid\theta) = \prod_{j=1}^{t} p(r_j\mid\sigma^2_j, \theta) \prod_{j=1}^{t} p(\sigma^2_j\mid\theta) \prod_{j=1}^{t} p(\theta) \tag{6}
\]

where

\[
p(r_j\mid\sigma^2_j, \theta) = \int p(r_j\mid\sigma^2_j, \theta) \, d\sigma^2_j 
\]

is obtained because, conditional to \( \sigma^2_j \), \( r_t \) is independent of \( r_T \) and \( \sigma^2 \) \( \forall \tau \neq t \), and

\[
p(\sigma^2_j\mid\theta) = \prod_{j=1}^{t} p(\sigma^2_j\mid\sigma^2_{j-1}, \theta),
\]

since \( \sigma^2_t \) is a Markov process as defined in (4).

The likelihood functions, calculated based on the distributions of the log-SV (2)-(3) and uGARCH Eqs. (4)-(5) models, are respectively given by

\[
L^{log} = \prod_{j=1}^{t} \mathcal{N}\left( r_j; 0, \log \frac{V_j}{2} \right) \mathcal{N}(V_j; \gamma + \phi V_{j-1}, \sigma) \, dV_{1:t} 
\]

\[
L^{uG} = \prod_{j=1}^{t} \mathcal{N}(r_j; 0, \sigma^2_j)^2 \mathcal{N}(\frac{\sigma^2_j - \omega - \beta \sigma^2_{j-1}}{\alpha \sigma^2_{j-1}}, \sigma) \, d\sigma^2_{1:t},
\]

where \( \mathcal{N}(.; 0, \sigma^2) \) denotes evaluation of the Gaussian density with zero mean and variance \( \sigma^2 \), and \( \chi^2(\cdot) \) denotes evaluation of the Chi-square distribution.

The SVMs likelihood functions in the above equations are defined as intractable integrals, which hinder the use of maximum likelihood for model fitting. Alternative approaches for parameter identification include expectation maximisation [20]; Monte Carlo maximum likelihood [21]; and [22], which states that Kalman filter-based techniques are “relatively more efficient” than the generalised method of moments introduced by [23].

For the particular case of the uGARCH model, recall that the GARCH likelihood function has a much simpler form due to the assumption of known volatility (conditional to the current observation), hence not requiring the integral expression in Eq. (6), since the Chapman-Kolmogorov formula is not used. The work in [15] takes advantage of this concept and, for convenience, implements a uGARCH model with parameters corresponding standard GARCH. Even though this approach provided accurate estimates compared to those of the offline (batch) GARCH, its heuristic approach does not guarantee optimality nor suitability for nonstationary scenarios.

IV. Time-varying Volatility Estimation

As pointed out in the previous section, a drawback of stochastic volatility models is the intractable integral form of the likelihood functions which burdens the task of parameter identification. Additionally, even if a suitable set of parameters is obtained using numerical methods, these will only be representative of the training data used to fit the models and might not provide reliable estimates when the behaviour of the market changes. We rectify this by allowing the parameters of stochastic volatility models to be time-varying and part of the state of the model, this way, the proposed model is well-suited for nonstationary scenarios and their parameters are estimated using nonlinear techniques such as particle filters.

The joint estimation of state and parameters, represented by the calculation of the posterior density \( p(\sigma_t, \theta_t|r_{1:t}) \), is known as self-organising state-space models [24], [11] and it is well known to provide reasonable estimates when the parameters are allowed to change gradually. To this end, the evolution of \( \theta_t \) is defined using Gaussian perturbations of the form [10]

\[
\theta_t = \theta_{t-1} + \eta_t^\theta, \quad \eta_t^\theta \sim \mathcal{N}(0, \sigma_t^\theta),
\]

where \( \sigma^\theta \) is a small standard deviation, thus allowing for low parameter noise but at the same time-varying capability. Examples where \( \sigma^\theta \) is manipulated, on the basis of a prediction performance measure, can be found in [25].

Using this concept, the time-varying extension of the log-SV model can be expressed as

\[
\begin{bmatrix}
V_{t+1} \\
\gamma_{t+1} \\
\phi_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
\gamma_t + \phi_t V_t + \eta_t \\
\gamma_t + \eta_t^\gamma \\
\phi_t + \eta_t^\phi
\end{bmatrix} \nonumber
\]

\[
r_t = \exp\left( \frac{1}{2} V_t \right) \epsilon_t, \quad \epsilon_t \sim \chi^2(\cdot),
\]

whereas the time-varying uGARCH takes the form

\[
\begin{bmatrix}
\sigma^2_{t+1} \\
\alpha_{t+1} \\
\beta_{t+1} \\
\mu_{t+1} \\
\omega_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
\omega_t + \alpha_t \sigma^2_t + \beta_t \sigma^2_t \eta_t^\omega \\
\alpha_t + \eta_t^\alpha \\
\beta_t + \eta_t^\beta \\
\mu_t + \eta_t^\mu \\
\omega_t + \eta_t^\omega
\end{bmatrix}
\]

In terms of implementation complexity, the main difference between the time-varying SVM models in Eqs. (7)-(8) and (9)-(10), and their time-invariant counterparts in Eqs. (2)-(3) and (4)-(5) respectively, is that the former feature highly coupled higher dimensional states, and may suffer from the curse of dimensionality. In this sense, we suggest to use particle filters for the joint estimation of state and parameters, since this class of filters have a rate of convergence which is independent of the state dimension [26].

Observe that although Eqs. (7)-(10) consider all model parameters to be time-varying, in practice some of them may remain fairly constant. We suggest to consider a preliminary assessment of the algorithm to discard unnecessary time-varying parameters (and therefore state dimension), in order...
to focus the computational resources towards achieving more accurate estimates, e.g. by increasing the number of particles of the particle filter [9].

V. Simulations

The proposed time-varying extensions of the log-SV and uGARCH models were implemented alongside the sample importance resample particle filter (SIR) [27] for (i) estimating the volatility of two stock market indices, and (ii) adaptively identifying the model parameters. The aim of our simulations is to show that the proposed online algorithms (time-varying SVMs together with the SIR filter) achieve levels of accuracy similar to those of offline (batch) algorithms that consider all observations for a given period. The proposed models were quantitatively assessed in terms of the estimation error.

The stock market indices considered were the NASDAQ Composite (NASDAQ-C) and the General Stock Price Index (Indice General de Precios de Acciones, or IGPA) published by the Santiago stock exchange (SSE) in Chile. The NASDAQ-C is an indicator of all the components listed in the NASDAQ stock market, and it is highly followed as an indicator of technology companies, whereas the IGPA is a market capitalisation-weighted index that measures price variations of the majority of the stocks traded in the SSE. The motivation for analysing stock market indices stems from their weighted average nature, which comprises several sources of uncertainty.

A. Data preprocessing

The returns \( r_t \) corresponding to the observed daily closing prices \( p_t \) were first calculated by [28]

\[
r_t = \log \left( \frac{p_{t+1}}{p_t} \right),
\]

and then analysed to justify the use of volatility models. Fig. 1 shows the sample correlation of the squared returns \( r_t^2 \) for each index, together with the 95% confidence interval where the theoretical autocorrelation function is expected to have vanished. Two distinguishing features can be observed from this figure: (i) the decaying correlation of the squared returns suggest the use of autoregressive volatility models, and (ii) the NASDAQ-C correlation decays very slowly, revealing the nonstationary behaviour of the signal, a common factor in financial time series.

The median and first four moments of the return series are shown in Table I, where the high kurtosis (greater than three) reveal the fat tail property of the returns distribution.

### Table I: Sample statistics for the indices considered (November 2002- September 2010). The skewness and kurtosis were standardised with respect to the standard deviation.

<table>
<thead>
<tr>
<th>Index</th>
<th>Median</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>NASDAQ-C</td>
<td>8.89e-4</td>
<td>2.86e-4</td>
<td>2.21e-4</td>
<td>-0.13</td>
<td>9.1</td>
</tr>
<tr>
<td>IGPA</td>
<td>9.8e-4</td>
<td>7.6e-4</td>
<td>6.6e-5</td>
<td>-0.07</td>
<td>15.09</td>
</tr>
</tbody>
</table>

The period considered for our simulations comprised 635 samples, from June 2007 to January 2010, and included the start of the late-2000s recession (September 2008, according to the National Bureau of Economic Research).

B. Volatility estimation using the time-varying log-SV

A preliminary analysis, based on the proposed time-varying log-SV model in Eqs. (7)-(8), indicated that only the correlation parameter \( \phi \) needed to be recursively estimated. The considered model was

\[
\begin{align*}
V_{t+1} &= \gamma + \phi_t V_t + \eta_t \\
\phi_{t+1} &= \phi_t + \eta_t^\phi \\
r_t &= \exp \left( \frac{1}{2} V_t \right) \epsilon_t,
\end{align*}
\]

where the (fixed) model parameters and initial conditions were set by assessing the performance of the algorithms during the first 250 samples of the considered period (June 2007 - June 2008), and are shown in Table III in the Appendix.

For the NASDAQ-C and the IGPA respectively, Figs. 2 and Fig. 3 show—from top to bottom—the offline GARCH and online log-SV volatility estimates, the time-varying parameter \( \phi \), and the return series.

Observe that by only using data until \( t = 250 \) for training, the log-SV algorithm was capable of tracking the volatility peak corresponding to the late-2000s crisis around sample \( t = 320 \) for both indices. Accordingly, note that \( \phi_t \) decreased during this period (see the red circles), thus meaning that the volatility was driven by the stochastic innovation rather than the deterministic (autoregressive) part of the model [see Eq. (2)].

![Fig. 1: Correlation coefficients for the return series of both IGPA and NASDAQ-C indices (November 2002 - September 2010).](image-url)

![Fig. 2: Volatility estimation using the time-varying log-SV model for NASDAQ-C](image-url)

![Fig. 3: Volatility estimation using the time-varying log-SV model for IGPA](image-url)
C. Volatility estimation using the time-varying uGARCH

Based on a preliminary implementation of the uGARCH model in Eqs. (9)-(10) for the estimation of the aforementioned indices volatility, only parameters $\alpha$ and $\beta$ were recursively estimated, thus yielding the model

$$
\begin{bmatrix}
\sigma_{t+1}^2 \\
\alpha_{t+1} \\
\beta_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\omega + \alpha_t \sigma_t^2 + \beta_t \sigma_t^2 \eta_t^2 \\
\alpha_t + \eta_t^\alpha \\
\beta_t + \eta_t^\beta
\end{bmatrix}
\begin{bmatrix}
r_t
\end{bmatrix}
= 
\begin{bmatrix}
\mu + \sigma_t \epsilon_t
\end{bmatrix}
$$

The fixed model parameters chosen based on the first 250 samples (June 2007 - June 2008) are presented in Table IV.

Figs. 4 and 5 show the time-varying uGARCH volatility estimates, the parameters, and the return series for both indices. Observe that parameter estimates (middle plots) revealed that the parameter $\alpha_t$, related to the innovation process, raised considerably whenever there were volatility peaks (see the red circles), suggesting that the volatility was mainly driven by the innovation process rather than the autoregressive component [see Eq. (4)].
D. Performance assessment

The performance of the proposed time-varying SVMs was next analysed using a normalised error measure, whereby an offline (batch) GARCH estimate was used as ground truth signal. The GARCH smoothed estimate is used as benchmark since its estimate, $\hat{\sigma}_t$, considers all observations (past, present and future), and hence it is assumed to be better—in the sense of minimum squares—than an estimate based only on past observations (see the Rao-Blackwell theorem [29]).

The error measure considered is given by

$$E_\sigma = \frac{1}{T} \sum_{t=1}^{T} \frac{|\hat{\sigma}_t - \hat{\sigma}_t|}{\bar{\sigma}_t} \times 100$$

where $\hat{\sigma}_t$ is the estimate and $\bar{\sigma}_t$ is the ground truth. The factor 100 is used to express the error as a percentage of the ground truth, and small values for $E_\sigma$ indicate accurate estimates.

Table II shows the error associated to the proposed time-varying SVMs for both the NASDAQ-C and IGPA financial indices.

<table>
<thead>
<tr>
<th>Model</th>
<th>NASDAQ-C</th>
<th>IGPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time-varying log-SV</td>
<td>18.82%</td>
<td>17.35%</td>
</tr>
<tr>
<td>Time-varying uGARCH</td>
<td>7.86%</td>
<td>6.47%</td>
</tr>
</tbody>
</table>

The percentage error of the proposed uGARCH with time-varying parameters was around 6-7%, whereas the time-varying log-SV presents a higher percentage error, however, note that this is not a comparison between the log-SV and uGARCH models, since error measure $E_\sigma$ is calculated with respect to the batch GARCH estimate for both time-varying algorithms, which gives the log-SV a slight disadvantage.

The low percentage errors in Table II validate the suitability of the proposed models for reliable and online volatility tracking in nonstationary scenarios.

VI. CONCLUDING REMARKS

A class of stochastic volatility models with adaptive parameter identification has been proposed and evaluated for online estimation of the volatility of financial returns series. The presented algorithm models both the volatility and the time-varying parameters as states of a hidden Markov model, thus enabling the use of nonlinear filters to jointly estimate the volatility and the model parameters. Simulations on online adaptive estimation of two financial indices, the NASDAQ-C and the Chilean IGPA, have validated the proposed framework and revealed that: (i) the proposed time-varying SVMs achieved levels of accuracy comparable to those of offline (batch) algorithms but required fewer training samples and are well-suited for online applications, (ii) the model identification provided physical meaning and captured market changes at early stages in the form of sudden parametric changes, and (iii) the particle filter stage of the proposed algorithms allowed for the estimation of confidence intervals.

The accuracy, parametric identification ability, and statistical description properties of the proposed algorithm validate its suitability as an online volatility estimator that can be used for market analysis or portfolio design, where the estimation of the volatility needs to be accurate, online and adaptive.

APPENDIX

The initial conditions of the time-varying parameters, as well as the fixed parameters, were chosen by assessing the performance of the algorithms throughout the first 250 samples (June 2007 to June 2008). The innovation processes driving the time-varying parameters were zero-mean Gaussians with a standard deviation corresponding to 0.01 times the initial conditions. These values are presented in the following tables.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>NASAQ-C</th>
<th>IGPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$-4.19 \times 10^{-1}$</td>
<td>$-5.99 \times 10^{-1}$</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>$9.51 \times 10^{-1}$</td>
<td>$9.36 \times 10^{-1}$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$8.48 \times 10^{-2}$</td>
<td>$2.35 \times 10^{-1}$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$9.51 \times 10^{-3}$</td>
<td>$9.36 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Parameter</th>
<th>NASAQ-C</th>
<th>IGPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega$</td>
<td>$8 \times 10^{-6}$</td>
<td>$6.89 \times 10^{-6}$</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$2.44 \times 10^{-4}$</td>
<td>$7.21 \times 10^{-4}$</td>
</tr>
<tr>
<td>$\alpha_0$</td>
<td>$8.91 \times 10^{-1}$</td>
<td>$7.38 \times 10^{-1}$</td>
</tr>
<tr>
<td>$\beta_0$</td>
<td>$6.77 \times 10^{-2}$</td>
<td>$2.06 \times 10^{-1}$</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>$8.91 \times 10^{-3}$</td>
<td>$7.38 \times 10^{-3}$</td>
</tr>
<tr>
<td>$\sigma_1$</td>
<td>$6.77 \times 10^{-4}$</td>
<td>$2.06 \times 10^{-3}$</td>
</tr>
</tbody>
</table>

REFERENCES


