Estimation of Financial Indices Volatility Using a Model with Time-Varying Parameters

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Outline

- How to design volatility models?
- The GARCH model
- Stochastic volatility model
- GARCH-based SV model
- SVM parametric identification using particle filters
- Simulations and conclusions
How do we model volatility?

The term *volatility* $\sigma_t$ refers to the standard deviation of the variation (return) of a financial instrument, that is,

$$r_t = \mu + \sigma_t \epsilon_t$$

where $\mu$ is the expected return and $\epsilon_t$ is a unit-variance process.

A volatility model needs to account for:

- Heteroscedasticity and volatility clustering
- Right-skewed and leptokurtic return distributions.
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A volatility model needs to account for:

- Heteroscedasticity and volatility clustering
- Right-skewed and leptokurtic return distributions.
- Can we provide full statistical description of the volatility?
- Can we guarantee these properties for a time-varying model?
- How should the model vary? i.e. how do we choose the model parameters?
Generalized AutoRegressive Conditional Heteroskedasticity (GARCH)

Proposed by Tim Bollerslev, 1986:

\[
\sigma_{t|t-1}^2 = \omega + \alpha u_{t-1}^2 + \beta \sigma_{t-1|t-2}^2 \\
\]

\[
r_t = \mu + u_t.
\]

✓ Parametric identification in closed form: \( L(\theta| r_{1:T}, \Sigma_0) \rightarrow \) is tractable (ML).

✓ Straightforward to implement

✗ The GARCH model computes the volatility in a deterministic fashion as an autoregressive process driven by the observed return.
Stochastic volatility model

Let us assume a state space model for the volatility (state) and the returns (observation), that is,

\[
\begin{align*}
\sigma_t &= f(\sigma_{t-1}, \theta) + \eta_t \\
rt &= g(\sigma_t, \theta)\epsilon_t
\end{align*}
\]

where \(\eta_t, \epsilon_t\) and noise processes, \(f, g\) are known functions and \(\theta\) is a set of unknown parameters.

✓ Combined with sequential Monte Carlo methods, provides the full posterior density \(p(\sigma_{0:t}|r_{1:t})\)

\[
L(\theta|r_{1:T}) = \int \prod_{j=1}^{t} p(r_j|\sigma_j, \theta)p(\sigma_j|\sigma_{j-1}, \theta)d\sigma_{1:t} \rightarrow \text{is intractable.}
\]
Proposal: A GARCH-based SV model

The unobserved GARCH (uGARCH) is given by

\[ \sigma_t^2 = \omega + \alpha \sigma_{t-1}^2 \eta_{t-1}^2 + \beta \sigma_{t-1}^2 \]
\[ r_t = \mu + \sigma_t \epsilon_t. \]

where \( \epsilon_t, \eta_t \) are noise processes.

How do we find the model parameters?

- Expectation-Maximisation (offline)
- Use the GARCH parameters (offline)
- Turn the parameters into “states” and use filtering algorithms (online)
Adaptive identification using particle filters

Assume the unknown parameters evolve \textit{artificially} according to a random walk $\theta_t = \theta_{t-1} + \eta^\theta_t$ and consider them as state, that is,

\[
\begin{bmatrix}
\sigma^2_{t+1} \\
\alpha_{t+1} \\
\beta_{t+1} \\
\mu_{t+1} \\
\omega_{t+1}
\end{bmatrix}
= 
\begin{bmatrix}
\omega_t + \alpha_t \sigma^2_t + \beta_t \sigma^2_t \eta^2_t \\
\alpha_t + \eta^\alpha_t \\
\beta_t + \eta^\beta_t \\
\mu_t + \eta^\mu_t \\
\omega_t + \eta^\omega_t
\end{bmatrix}
\]

\[r_t = \mu_t + \sigma_t \epsilon_t.\]
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✓ Nonlinear state space model $\Rightarrow$ use \textbf{particle filters}
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\end{bmatrix}
\]

\[r_t = \mu_t + \sigma_t \epsilon_t.\]

✓ Nonlinear state space model ⇒ use particle filters
✓ Possible due to the reduced size of the parameter space
Adaptive identification using particle filters

Assume the unknown parameters evolve artificially according to a random walk $\theta_t = \theta_{t-1} + \eta_t^\theta$ and consider them as state, that is,

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\end{bmatrix}
$$

$$r_t = \mu_t + \sigma_t \epsilon_t.$$

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- Allows for time-varying parameters
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\]

$\mathbf{r}_t = \mu_t + \sigma_t \epsilon_t$.

✓ Nonlinear state space model ⇒ use \textbf{particle filters}
✓ Possible due to the reduced size of the parameter space
✓ Allows for \textbf{time-varying} parameters
✓ \textbf{Artificial evolution} is the naive approach, it can be improved!
Adaptive identification using particle filters

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\end{bmatrix}
\]

\[r_t = \mu_t + \sigma_t \epsilon_t.\]

✓ Nonlinear state space model ⇒ use particle filters
✓ Possible due to the reduced size of the parameter space
✓ Allows for time-varying parameters
✓ Artificial evolution is the naive approach, it can be improved!
✓ Not all the parameters need to be time varying
Simulations on two financial indices: NASDAQ-C and Chilean IGPA

Autocorrelation function of square returns (2002-2010)

Sample statistics of returns (2002-2010)

<table>
<thead>
<tr>
<th>Index</th>
<th>Median</th>
<th>Mean</th>
<th>Variance</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>NASDAQ-C</td>
<td>8.89e-4</td>
<td>2.86e-4</td>
<td>2.21e-4</td>
<td>-0.13</td>
<td>9.1</td>
</tr>
<tr>
<td>IGPA</td>
<td>9.8e-4</td>
<td>7.6e-4</td>
<td>6.6e-5</td>
<td>-0.07</td>
<td>15.09</td>
</tr>
</tbody>
</table>
Joint filtering of volatility and model parameters

Period: June 2007 to January 2010
Model: uGARCH + particle filter (only $\alpha$ and $\beta$ vary)
Performance assessment for uGARCH

We considered the error measure

\[
E_{\hat{\sigma}} = \frac{1}{T} \sum_{t=1}^{T} \frac{|\bar{\sigma}_t - \hat{\sigma}_t|}{\bar{\sigma}_t} \times 100
\]

where

- \( \hat{\sigma}_t \) is the online uGARCH estimate
- \( \bar{\sigma}_t \) is the batch GARCH estimate

<table>
<thead>
<tr>
<th>Model</th>
<th>NASDAQ-C</th>
<th>IGPA</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time-varying uGARCH</td>
<td>7.86%</td>
<td>6.47%</td>
</tr>
</tbody>
</table>

✓ Comparable accuracy with online capability
Artificial evolution with the log-SV model

The log-SV model, together with the parameters equations, is given by

\[
\begin{bmatrix}
V_{t+1} \\
\phi_{t+1}
\end{bmatrix} =
\begin{bmatrix}
\gamma + \phi_t V_t + \eta_t \\
\phi_t + \eta_t^\phi
\end{bmatrix}
\]

\[r_t = \exp\left(\frac{1}{2} V_t\right) \epsilon_t,\]
Joint filtering of volatility and model parameters

**Period:** June 2007 to January 2010

**Model:** log-SV + particle filter (only $\phi$ varies)

- **a)** 95% confidence interval, log-SV estimate, GARCH (Offline)
- **b)** Online parameter estimate
- **c)** IGPA returns

**NASDAQ-C**
Performance assessment for log-SV

We considered the error measure

\[ E_{\hat{\sigma}} = \frac{1}{T} \sum_{t=1}^{T} \frac{|\bar{\sigma}_t - \hat{\sigma}_t|}{\bar{\sigma}_t} \times 100 \]

where

- \( \hat{\sigma}_t \) is the **online** log-SV estimate
- \( \bar{\sigma}_t \) is the **batch** GARCH estimate

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<tr>
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<th>NASDAQ-C</th>
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</tr>
</thead>
<tbody>
<tr>
<td>Time-varying log-SV</td>
<td>18.62%</td>
<td>17.55%</td>
</tr>
</tbody>
</table>

✓ Note that the comparison is with respect to the GARCH batch estimate
Discussion.

- Meaningful parameter estimates (which do not require additional constraints such as $\alpha + \beta < 1$)
- Parameter changes allow for early anomaly detection (analogy with fault detection)
- Joint filtering, prediction, smoothing, and parameter identification using particle filters
- Full statistical description (higher order moments and confidence intervals)
- Online capability and as accurate as its batch counterpart
Discussion.

- Meaningful parameter estimates (which do not require additional constraints such as \( \alpha + \beta < 1 \))
- Parameter changes allow for early *anomaly* detection (analogy with fault detection)
- Joint filtering, prediction, smoothing, and parameter identification using particle filters
- Full statistical description (higher order moments and confidence intervals)
- Online capability and as accurate as its batch counterpart

- PF is prone to particle degeneracy in not resampled properly
- The evolution step can be improved (Gaussian process, support vector machines, expert knowledge, etc)
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