The negative binomial
Indian buffet process

Creighton Heaukulani
University of Cambridge

Joint work with Daniel M. Roy (Cambridge)

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Outline

1. The Indian buffet process (IBP; [GG06, GGS07]) induces a distribution on allocations of features to individuals. We are interested in count extensions of the IBP, i.e., each individual may have multiple copies of each feature.

2. ([TJ07]). The IBP is the combinatorial structure of an exchangeable sequence of Bernoulli processes directed by a beta process base measure.

3. We are interested in the combinatorial structure of an exchangeable sequence of negative binomial processes ([BMPJ11, ZHDC12]), directed by a beta process, which we describe as the negative binomial Indian buffet process, a count extension of the IBP.
Plan

1. Review the IBP and connect it to the theory of completely random measures.

2. Develop a different continuum of Pólya urn schemes perspective for sampling the IBP.

3. Develop a method to sample a beta negative binomial process using the continuum of Pólya urn schemes intuition.

4. Present the corresponding negative binomial IBP.
Indian buffet process

Let $\alpha, c > 0$. A sequence of customers go through an Indian buffet with an infinite number of dishes.
Indian buffet process

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- The first customer $n = 1$ tastes $\text{Poisson}(\alpha)$ dishes;
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- The $n+1$-st customer:
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- The first customer $n = 1$ tastes $\text{Poisson}(\alpha)$ dishes;

- The $n + 1$-st customer:
  - takes $\text{Bernoulli}\left(\frac{m_k}{c+n}\right)$ servings of every previously sampled dish $k$, where $m_k$ is the total number of servings taken of dish $k$ by the previous $n$ customers;
Indian buffet process

Let $\alpha, c > 0$. A sequence of customers go through an Indian buffet with an infinite number of dishes.

- The first customer $n = 1$ tastes $\text{Poisson}(\alpha)$ dishes;

- The $n + 1$-st customer:
  - takes $\text{Bernoulli}\left(\frac{m_k}{c+n}\right)$ servings of every previously sampled dish $k$, where $m_k$ is the total number of servings taken of dish $k$ by the previous $n$ customers;
  - tastes $\text{Poisson}(\alpha \frac{c}{c+n})$ new dishes.
**IBP: matrix perspective**

rows = individuals/customers  
columns = features/dishes

<table>
<thead>
<tr>
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</table>
IBP: matrix perspective

rows = individuals/customers
columns = features/dishes

\[
\begin{array}{ccc}
  n = 1 & 1 & 1 \\
  n = 2 & 1 & 1 & 1 \\
  \vdots & 1 & 1 & 1 \\
  \vdots & 1 & 1 & 1 & 1 \\
\end{array}
\]

[TJ07] connected the IBP with the theory of completely random measures.
IBP: matrix perspective

rows = individuals/customers Bernoulli processes
columns = features/dishes locations in $\Omega$

\[
\begin{array}{cccccc}
\omega_1 & \omega_2 & \omega_3 & \omega_4 & \cdots \\
X_1 & 1 & 1 & & & \\
X_2 & & 1 & 1 & 1 & \\
X_3 & 1 & 1 & 1 & & \\
X_4 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

[TJ07] connected the IBP with the theory of completely random measures.
IBP: matrix perspective

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columns = features/dishes locations in $\Omega$

$[TJ07]$ connected the IBP with the theory of completely random measures.
IBP: matrix perspective

rows = individuals/customers Bernoulli processes
columns = features/dishes locations in $\Omega$

\[ X_{n+1} | X_1, \ldots, X_n \sim \text{BeP}(c + n, \tilde{B}_0 + c + n \sum_{i=1}^{X_i}) \]

[TJ07] connected the IBP with the theory of completely random measures.
IBP: matrix perspective

rows = individuals/customers Bernoulli processes
columns = features/dishes locations in Ω

\[ X_{n+1} | X_1, \ldots, X_n \sim \text{BeP} \left( c + n, \tilde{B}_0 + 1 \right) \]

\[ X_2 = \delta_{\omega_2} + \delta_{\omega_3} + \delta_{\omega_4} \]

[TJ07] connected the IBP with the theory of completely random measures.
IBP: matrix perspective

rows = individuals/customers Bernoulli processes
columns = features/dishes locations in $\Omega$

$$X_{n+1} \mid X_1, \ldots, X_n \sim \text{BeP} \left( \frac{c}{c+n} \tilde{B}_0 + \frac{1}{c+n} \sum_{i=1}^{n} X_i \right)$$

$$X_2 = \delta_{\omega_2} + \delta_{\omega_3} + \delta_{\omega_4}$$

[TJ07] connected the IBP with the theory of completely random measures.
Let \((X_n)_{n \in \mathbb{N}}\) satisfy

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\]
IBP: completely random measures perspective

Let \((X_n)_{n \in \mathbb{N}}\) satisfy

\[
X_{n+1} | X_1, \ldots, X_n \sim \text{BeP}\left(\frac{c}{c+n} \tilde{B}_0 + \frac{1}{c+n} \sum_{i=1}^{n} X_i\right).
\]

**Thm ([TJ07]):** Then there exists a beta process

\[
B \sim \text{BP}(c, \tilde{B}_0)
\]

such that

\[
(X_n)_{n \in \mathbb{N}} | B \overset{iid}{\sim} \text{BeP}(B).
\]
How to get counts

$$(X_n)_{n \in \mathbb{N}} \mid B \overset{iid}{\sim} \text{BeP}(B)$$

\[\begin{array}{cccc}
  n = 1 & 1 & 1 \\
  n = 2 & 1 & 1 & 1 \\
  \vdots & 1 & 1 & 1 \\
  \vdots & 1 & 1 & 1 & 1
\end{array}\]
How to get counts

\[(X_n)_{n \in \mathbb{N}} \mid B \overset{iid}{\sim} \text{BeP}(B)\]

Bernoulli(\(p\)) \rightarrow \text{geometric}(p): \# \text{ successes before the first failure}
How to get counts

$$(X_n)_{n \in \mathbb{N}} \ | \ B \overset{iid}{\sim} \text{BeP}(B)$$

Bernoulli($p$) $\rightarrow$ geometric($p$): $\#$ successes before the first failure

<table>
<thead>
<tr>
<th>$\omega_1$</th>
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<th>$\omega_4$</th>
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$\delta_{\omega_1} + \cdots$
How to get counts

$$(X_n)_{n \in \mathbb{N}} \mid B \overset{iid}{\sim} \text{BeP}(B)$$

Bernoulli($p$) $\rightarrow$ geometric($p$): # successes before the first failure

$\delta_{\omega_1} + 2\delta_{\omega_2} + \ldots$

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How to get counts

\[(X_n)_{n \in \mathbb{N}} \mid B \overset{iid}{\sim} \text{BeP}(B)\]

Bernoulli(\(p\)) \(\rightarrow\) geometric(\(p\)): \# successes before the first failure

\[\delta_{\omega_1} + 2 \delta_{\omega_2}\]
How to get counts

$$(X_n)_{n \in \mathbb{N}} | \quad B \overset{iid}{\sim} \operatorname{BeP}(B)$$

Bernoulli($p$) $\rightarrow$ geometric($p$): \# successes before the first failure

$$\delta_{\omega_1} + 2 \delta_{\omega_2} \sim \operatorname{BGP}(\tilde{B}_0)$$
How to get counts

$$(X_n)_{n \in \mathbb{N}} \mid B \overset{iid}{\sim} \text{BeP}(B)$$

Bernoulli($p$) $\rightarrow$ geometric($p$): # successes before the first failure

\[
\begin{array}{cccccc}
\omega_1 & \omega_2 & \omega_3 & \omega_4 & \cdots \\
X_1 & 1 & 1 & & & \\
X_2 & 1 & 1 & 1 & & \\
X_3 & 1 & 1 & 1 & & \\
X_4 & 1 & 1 & 1 & 1 & \\
\end{array}
\]

$\delta_{\omega_1} + 2 \delta_{\omega_2} \sim \text{BGP}(\tilde{B}_0)$

NB($r, p$): # successes before the $r$-th failure
Imagine an infinite number of independent Pólya urn schemes, each with two tables labelled 1 and 0.
Imagine an infinite number of independent Pólya urn schemes, each with two tables labelled 1 and 0
Urn scheme perspective

Imagine an infinite number of independent Pólya urn schemes, each with two tables labelled 1 and 0

i.e., a continuum of Pólya urn schemes
Urn scheme perspective

How to sample $X_1 \sim \text{BeP}(\tilde{B}_0)$
Urn scheme perspective

How to sample $X_1 \sim \text{BeP}(\tilde{B}_0)$

- A Poisson($\alpha$) number of urn schemes succeed.

\[ \begin{array}{c@{}c@{}c}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} \]

\[ \Omega \]
Urn scheme perspective

How to sample $X_1 \sim \text{BeP}(\tilde{B}_0)$

- A Poisson($\alpha$) number of urn schemes succeed.

$\tilde{B}_0(\Omega) = \alpha$
Urn scheme perspective

How to sample $X_1 \sim \text{BeP}(\tilde{B}_0)$

- A Poisson($\alpha$) number of urn schemes succeed.
- The remaining urn schemes on the continuum fail.
Urn scheme perspective

How to sample $X_1 \sim \text{BeP}(\tilde{B}_0)$

- A Poisson($\alpha$) number of urn schemes succeed.
- The remaining urn schemes on the continuum fail.

$$X_1\{\omega_1\} = 1 \quad X_1\{\omega_2\} = 1$$

$X_1$ done.
How to sample $X_2 \mid X_1$

$$X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1} \tilde{B}_0 + \frac{1}{c+1} X_1\right)$$
How to sample $X_2 \mid X_1$

Run the urn schemes forward one more step

$$X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1} \tilde{B}_0 + \frac{1}{c+1} X_1\right)$$
How to sample $X_2 \mid X_1$

Run the urn schemes forward one more step
⇒ Consider the previous atoms in $X_1$

\[
X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1} \tilde{B}_0 + \frac{1}{c+1} X_1\right)
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How to sample $X_2 \mid X_1$

Run the urn schemes forward one more step
⇒ Consider the previous atoms in $X_1$
⇒ Consider urns not in $X_1$

\[ X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1} \tilde{B}_0 + \frac{1}{c+1} X_1\right) \]
How to sample $X_2 \mid X_1$

Run the urn schemes forward one more step
⇒ Consider the previous atoms in $X_1$
⇒ Consider urns **not** in $X_1$; a Poisson($\alpha \frac{c}{c+1}$) new urns succeed

\[
X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1} \tilde{B}_0 + \frac{1}{c+1} X_1\right)
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Run the urn schemes forward one more step
⇒ Consider the previous atoms in $X_1$
⇒ Consider urns not in $X_1$; a Poisson($\alpha \frac{c}{c+1}$) new urns succeed
⇒ The remaining urns on the continuum fail

$$X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1} \tilde{B}_0 + \frac{1}{c+1} X_1\right)$$
How to sample $X_2 \mid X_1$

Run the urn schemes forward one more step
⇒ Consider the previous atoms in $X_1$
⇒ Consider urns **not in** $X_1$; a $\text{Poisson}(\alpha \frac{c}{c+1})$ new urns succeed
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$$X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1} \tilde{B}_0 + \frac{1}{c+1} X_1\right)$$
How to sample $X_2 \mid X_1$

$X_2$ done.

$$X_2 \mid X_1 \sim \text{BeP}\left(\frac{c}{c+1}\tilde{B}_0 + \frac{1}{c+1}X_1\right)$$
A count extension

Bernoulli\((p)\): distribution of successful trial with success probability \(p\);

\(NB(r, p)\): distribution of \# successes until \(r\) failures.
A count extension

**Bernoulli**($p$): distribution of successful trial with success probability $p$;

**NB**($r, p$): distribution of # successes until $r$ failures.

**Intuition:** The Bernoulli process runs one trial at every urn scheme. So the *negative binomial process* continues running trials until $r$ failures at each urn scheme.

**Key point:** Instead of binary indicators at each urn scheme, we get integer-valued counts.
Negative binomial process
Negative binomial process

Negative binomial processes are parameterized by a “base measure”

\[
B_0 = \tilde{B}_0 + \sum_{k=1}^{\kappa} b_k \delta_{\omega_k}.
\] (1)
Negative binomial process

Negative binomial processes are parameterized by a “base measure”

\[ B_0 = \tilde{B}_0 + \sum_{k=1}^{\kappa} b_k \delta_{\omega_k}. \]  

(1)

**Defn.** We call \( X \sim \text{NBP}(r, B_0) \) a **negative binomial process**, when it is a completely random measure with

- fixed component (as defined by [BMPJ11, ZHDC12]):

\[ \sum_{k=1}^{\kappa} \zeta_k \delta_{\omega_k}, \quad \zeta_k^{\text{ind}} \sim \text{NB}(r, b_k), \quad \kappa \in \mathbb{N} \cup \{\infty\}, \]  

(2)
Negative binomial process

Negative binomial processes are parameterized by a "base measure"

\[ B_0 = \tilde{B}_0 + \sum_{k=1}^{\kappa} b_k \delta_{\omega_k}. \]  

(1)

**Defn.** We call \( X \sim \text{NBP}(r, B_0) \) a **negative binomial process**, when it is a completely random measure with

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  \[ \sum_{k=1}^{\kappa} \zeta_k \delta_{\omega_k}, \quad \zeta_k \overset{iid}{\sim} \text{NB}(r, b_k), \quad \kappa \in \mathbb{N} \cup \{\infty\}, \]  

(2)

- ordinary component (our definition): a Poisson process with intensity \( r \tilde{B}_0 \).
Beta negative binomial process

Our goal: give a method to directly sample $X$ without representing the underlying beta process.

$X \sim \text{BNBP}(r,c,B_0)$ when there exists a beta process $B \sim \text{BP}(c,B_0)$ such that $X \mid B \sim \text{NBP}(r,B)$.

$\text{BMPJ11, ZHDC12}$ use finite approximations to the beta process in order to produce samples of $X$. We aim to develop methods that avoid this representation.
Beta negative binomial process

**Defn.** We call \( X \sim \text{BNBP}(r, c, B_0) \) a beta negative binomial process when there exists a beta process

\[
B \sim \text{BP}(c, B_0)
\]

such that

\[
X \mid B \sim \text{NBP}(r, B).
\]
Beta negative binomial process

**Defn.** We call $X \sim \text{BNBP}(r, c, B_0)$ a beta negative binomial process when there exists a beta process $B \sim \text{BP}(c, B_0)$ such that

$$X \mid B \sim \text{NBP}(r, B).$$

(3)

- The beta process has an infinite number of atoms w.p. one
Defn. We call $X \sim \text{BNBP}(r, c, B_0)$ a beta negative binomial process when there exists a beta process $B \sim \text{BP}(c, B_0)$ such that $X \mid B \sim \text{NBP}(r, B)$.

\[ (3) \]

- The beta process has an infinite number of atoms w.p. one
- [BMPJ11, ZHDC12] use finite approximations to the beta process in order to produce samples of $X$
Beta negative binomial process

**Defn.** We call $X \sim \text{BNBP}(r, c, B_0)$ a beta negative binomial process when there exists a beta process

$$B \sim \text{BP}(c, B_0)$$

such that

$$X \mid B \sim \text{NBP}(r, B).$$

(3)

- The beta process has an infinite number of atoms w.p. one

- [BMPJ11, ZHDC12] use finite approximations to the beta process in order to produce samples of $X$

**Our goal:** give a method to directly sample $X$ without representing the underlying beta process.
Start simple

Consider a NB($r$, $p$) distribution when $r = 1$, i.e., a geometric distribution

\[ \text{NB}(1, p) = \text{geometric}(p), \]  

which counts the number of successful trials (with success probability $p$) before the **first failure**.
Start simple

Consider a NB\((r, p)\) distribution when \(r = 1\), i.e., a geometric distribution

\[
\text{NB}(1, p) = \text{geometric}(p),
\]

which counts the number of successful trials (with success probability \(p\)) before the first failure.

**Intuition:** Run the urn scheme until the first failure.
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

Run all urn schemes at once
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

Run all urn schemes at once
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

We continue until all urn schemes have failed once!
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

We continue until all urn schemes have failed once!

$\Rightarrow$ The remaining urns on the continuum have already failed
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

We continue until all urn schemes have failed once!

$\Rightarrow$ The remaining urns on the continuum have already failed
$\Rightarrow$ We **complete** the atoms
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

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How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

We continue until all urn schemes have failed once!

⇒ The remaining urns on the continuum have already failed
⇒ We **complete** the atoms
How to sample $X \sim \text{BGP}(c, \tilde{B}_0)$

$X \{\omega_1\} = 3$

$X \{\omega_2\} = 1$

$X \sim \text{BGP}(c, \tilde{B}_0)$
Ordinary component: BGP

**Thm** ([HR13]). Let $Y \sim \text{PP}(\tilde{B}_0)$ and let

$$\zeta_s \overset{\text{ind}}{\sim} \text{beta-geometric}(1, c), \quad s \in \Omega,$$

be independent from $Y$. Then

$$X = \sum_{s \in Y} (1 + \zeta_s) \delta_s \sim \text{BGP}(c, \tilde{B}_0).$$
Ordinary component: BGP

**Thm** ([HR13]). Let $Y \sim \text{PP}(\tilde{B}_0)$ and let

$$\zeta_s \overset{\text{ind}}{\sim} \text{beta-geometric}(1, c), \quad s \in \Omega,$$

completing the atom

be independent from $Y$. Then

$$X = \sum_{s \in Y} (1 + \zeta_s) \delta_s \sim \text{BGP}(c, \tilde{B}_0).$$
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

Continue running the urn schemes until $r$ failures!
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

⇒ complete the atoms

Continue running the urn schemes until $r$ failures!
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

$\Rightarrow$ complete the atoms

E.g., if $r = 2$, we stop.

Continue running the urn schemes until $r$ failures!
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

$\Rightarrow$ complete the atoms
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

$\Rightarrow$ complete the atoms
$\Rightarrow$ advance the remaining urns on the continuum
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

⇒ complete the atoms
⇒ advance the remaining urns on the continuum

- A Poisson\left(\alpha \frac{c}{c+1}\right) number of new urns succeed
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

$\Rightarrow$ complete the atoms

$\Rightarrow$ advance the remaining urns on the continuum

- A Poisson $\left(\alpha \frac{c}{c+1}\right)$ number of new urns succeed
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$\Rightarrow$ complete the atoms

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- A Poisson$\left(\alpha \frac{c}{c+1}\right)$ number of new urns succeed
- **Complete** the new atom
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

Start where $X \sim \text{BGP}(c, \tilde{B}_0)$ left off

⇒ complete the atoms
⇒ advance the remaining urns on the continuum

- A $\text{Poisson}(\alpha \frac{c}{c+1})$ number of new urns succeed
- Complete the new atom
How to sample $X \sim \text{BNBP}(r, c, \tilde{B}_0)$

$X \{\omega_1\} = 3$

$X \{\omega_3\} = 2$

$X \{\omega_2\} = 2$

$X \sim \text{BNBP}(r, c, \tilde{B}_0)$
Ordinary component: BNBP

Thm ([HR13]). Let

\[ Y_\ell \overset{ind}{\sim} \text{PP}(\frac{c}{c + \ell - 1} \tilde{B}_0), \quad \ell \in [r] = (1, \ldots, r), \tag{5} \]

and let

\[ \zeta_{\ell,s} \overset{ind}{\sim} \text{beta-NB}(r - \ell + 1, 1, c + \ell - 1), \quad \ell \in [r], \ s \in \Omega, \tag{6} \]

independent also from \((Y_r)\). Then

\[ X = \sum_{\ell=1}^{r} \sum_{s \in Y_\ell} (1 + \zeta_{\ell,s}) \delta_s \sim \text{BNBP}(r, c, \tilde{B}_0). \tag{7} \]
Ordinary component: BNBP

**Thm** ([HR13]). Let

\[ Y_\ell \overset{\text{ind}}{\sim} \text{PP}\left(\frac{c}{c + \ell - 1} \tilde{B}_0\right), \quad \ell \in [r] = (1, \ldots, r), \quad (5) \]

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\[ X = \sum_{\ell=1}^{r} \sum_{s \in Y_\ell} (1 + \zeta_{\ell,s}) \delta_s \overset{\text{BNBP}}{\sim} \text{BNBP}(r, c, \tilde{B}_0). \quad (7) \]

**Key point:** atoms arise in a sequence of \(r\) rounds.
A negative binomial IBP

Each customer $n \in \mathbb{N}$ goes through $r \in \mathbb{N}$ rounds at the buffet.
A negative binomial IBP

Each customer $n \in \mathbb{N}$ goes through $r \in \mathbb{N}$ rounds at the buffet.

- $n = 1$: Independently for each of $r$ rounds $\ell = 1, \ldots, r$,
  - tastes $\text{Poisson}\left(\alpha \frac{c}{c+\ell-1}\right)$ dishes;

- subsequent customers $n \geq 2$:
  - takes $\text{beta-NB}(r, m_k, c + (n-1)r)$ servings of each previously tasted dish $k$, where $m_k$ is the total number of servings taken by first $n$ customers;
  - independently for each of $r$ rounds $\ell = 1, \ldots, r$,
    - tries $\text{Poisson}\left(\alpha \frac{c}{c+\ell-1} + (n-1)r + \ell - 1\right)$ new dishes;
    - returns to each new dish for $\text{beta-NB}(r-\ell+1, 1, c + (n-1)r + \ell - 1)$ additional servings.
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Probability function

Let $\mathcal{H}_n \equiv \mathbb{Z}_+^n \backslash \{0^n\}$, and for $h \in \mathcal{H}_n$, let $M_h$ count the number of features $k$ where every customer $n$ has $h(n)$ copies of feature $k$. 
Probability function

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**Claim.** The probability function for the NB-IBP is given by

$$
\frac{\alpha^K N}{\prod_{h \in \mathcal{H}_N} M_h!} \exp \left( -\alpha \sum_{n=1}^{N} \sum_{\ell=1}^{r} \frac{c}{c + nr + \ell - 1} \right) \prod_{h \in \mathcal{H}_N} \left[ \frac{c}{c + Nr} \frac{\Gamma(s(h)) \Gamma(c + Nr + 1)}{\Gamma(c + Nr + s(h))} \right]^{M_h} \prod_{n \in \mathbb{N}} \binom{r + h(n) - 1}{r - 1}
$$

where $s(h) \equiv \sum_{n=1}^{N} h(n)$.
General $r > 0$

- The previous urn scheme was only valid for $r \in \mathbb{N}$.

- Urn schemes for general $r > 0$ reduces to the case of fractional $r \in (0, 1)$, which is desirable for some applications.

  - We use Poisson process calculus to reduce the $r$ rounds to one slightly more clever round.

  - Completions are no longer beta-NB variables, but what we call harmonic mixtures.
General $r > 0$

There is an analytical extension of the NB-IBP to general values of $r > 0$. The probability function looks similar:

**Claim** ([HR13]). Let $r > 0$. The probability function for the NB-IBP is given by

$$
\frac{(c \alpha)^{K_N}}{\prod_{h \in \mathcal{H}_N} M_h!} \exp \left( -c \alpha \sum_{n=1}^{N} \left[ \psi(c + (n + 1)r) - \psi(c + nr) \right] \right) \prod_{h \in \mathcal{H}_N} \left[ \frac{\Gamma(s(h)) \Gamma(c + Nr)}{\Gamma(c + Nr + s(h))} \prod_{n \in \mathbb{N}} \frac{\Gamma(r + h(n))}{h(n)! \Gamma(r)} \right]^{M_h}.
$$

(8)

**Key difference:** No longer a concept of rounds.
Extensions

We can also characterize the combinatorial structure of an exchangeable sequence of negative binomial processes, directed by a generalized beta process [Roy13], parametrized by a measurable family of EPPFs.
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A continuum of generalized Blackwell-MacQueen urn schemes
Extensions

Special cases include:

- hierarchies of beta processes [TJ07]
- stable beta processes [TG09, BJP12]
  - corresponding power-law NB-IBP
- hierarchies of stable beta processes
Probability function

**Claim.** The probability function for the NB-IBP is given by

\[
\frac{\alpha^K N}{\prod_{h \in \mathcal{H}_N} M_h !} \exp \left( -\alpha \sum_{n=1}^{N} \sum_{\ell=1}^{r} \frac{c}{c + nr + \ell - 1} \right) \prod_{h \in \mathcal{H}_N} \left[ \frac{c \Gamma(s(h)) \Gamma(c + Nr + 1)}{c + Nr \Gamma(c + Nr + s(h))} \right]^{M_h} \prod_{n \in \mathbb{N}} \binom{r + h(n) - 1}{r - 1}^{M_h}
\]
Probability function from a generalized beta

**Claim.** The probability function for the generalized NB-IBP is given by

\[
\frac{\alpha^{K_N}}{\prod_{h \in \mathcal{H}_N} M_h!} \exp \left( -\alpha \sum_{n=1}^{N} \sum_{\ell=1}^{r} \mathbb{P}(K_{nr+\ell} > K_{nr+\ell-1}) \right) \prod_{h \in \mathcal{H}_N} \left[ \mathbb{P}(K_{N_{r+1}} > K_{Nr}, Z_{N_{r+s(h)}} = \cdots = Z_{N_{r+1}}) \prod_{n \in \mathbb{N}} \binom{r + h(n) - 1}{r - 1} \right]^{M_h}
\]
Probability function from a generalized beta

**Claim.** The probability function for the generalized NB-IBP is given by

\[
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\(\mathbb{P}(K_{n+1} > K_n)\): probability of a new table at step \(n + 1\) ...

\(\mathbb{P}(K_{n+1} > K_n, Z_{n+m} = \cdots = Z_{n+1})\): probability of also staying there for next \(m\) steps
Beta processes, stick-breaking, and power laws.

T. Broderick, L. Mackey, J. Paisley, and M. Jordan.
Combinatorial clustering and the beta-negative binomial process, 2011, 1111.1802v2.

T. L. Griffiths and Z. Ghahramani.
Infinite latent feature models and the Indian buffet process.

Z. Ghahramani, T. Griffiths, and P. Sollich.
Bayesian nonparametric latent feature models.
See also the discussion and rejoinder.

N. L. Hjort.
Nonparametric Bayes estimators based on beta processes in models for life history data.

C. Heaukulani and D. M. Roy.
Urn schemes for the beta negative binomial process and the negative binomial Indian buffet process, 2013, Preprint.

Y. Kim.
Nonparametric Bayesian estimators for counting processes.

D. M. Roy.
The combinatorial structure underlying the beta process is described by a continuum of blackwell-macqueen urn schemes, 2013, Preprint.
Y. W. Teh and D. Görür.
Indian buffet processes with power-law behavior.

R. Thibaux and M. I. Jordan.
Hierarchical beta processes and the Indian buffet process.

Beta-negative binomial process and Poisson factor analysis.