The Infinite Factorial Hidden Markov Model

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1 Introduction

This appendix describes the computation when taking the limit $M \to \infty$ for the finite dynamic latent feature model. This result, together with more background on the mIBP and iFHMM will appear in a tech report on the author's website.

Given the results in the paper submission, we want to compute the limit $M \to \infty$ of the equivalence class of $\boldsymbol{S}$

$$\lim_{M\to\infty} p([\boldsymbol{S}]) = \lim_{M\to\infty} \sum_{\boldsymbol{S}\in[\boldsymbol{S}]} p(\boldsymbol{S}|\alpha,\gamma,\delta) \tag{1}$$

$$= \lim_{M\to\infty} \frac{M!}{\prod_{h=0}^{2^T-1} M_h!} \prod_{m=1}^{M} \frac{\frac{\alpha}{M}\Gamma(\frac{\alpha}{M}+c_m^{01})\Gamma(c_m^{00}+1)\Gamma(\gamma+\delta)\Gamma(\delta+c_m^{10})\Gamma(\gamma+c_m^{11})}{\Gamma(\frac{\alpha}{M}+c_m^{00}+c_m^{01}+1)\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+\delta+c_m^{10}+c_m^{11})} \tag{2}$$

In order to compute this limit, we need to distinguish between two types of Markov chains: those with all zero states ($c_m^{00} = T$) and those with nonzero states ($c_m^{00} < T$). Let us denote with M_0 the number of Markov chains with all zero states and with M_+ the number of Markov chains that which have nonzero states; note that $M = M_0 + M_+$. We can rewrite the product in the previous equation based on this dichotomy

$$\prod_{m=1}^{M} \frac{\frac{\alpha}{M}\Gamma(\frac{\alpha}{M}+c_m^{01})\Gamma(c_m^{00}+1)\Gamma(\gamma+\delta)\Gamma(\delta+c_m^{10})\Gamma(\gamma+c_m^{11})}{\Gamma(\frac{\alpha}{M}+c_m^{00}+c_m^{01}+1)\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+\delta+c_m^{10}+c_m^{11})} \tag{3}$$

$$= \left(\frac{\alpha\Gamma(\frac{\alpha}{M})\Gamma(T+1)}{M\Gamma(\frac{\alpha}{M}+T+1)}\right)^{M_0} \prod_{m=1}^{M_+} \frac{\frac{\alpha}{M}\Gamma(\frac{\alpha}{M}+c_m^{01})\Gamma(c_m^{00}+1)\Gamma(\gamma+\delta)\Gamma(\delta+c_m^{10})\Gamma(\gamma+c_m^{11})}{\Gamma(\frac{\alpha}{M}+c_m^{00}+c_m^{01}+1)\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+\delta+c_m^{10}+c_m^{11})} \tag{4}$$

$$= \left(\frac{\Gamma(\frac{\alpha}{M}+1)\Gamma(T+1)}{\Gamma(\frac{\alpha}{M}+T+1)}\right)^{M} \prod_{m=1}^{M_+} \frac{\Gamma(\frac{\alpha}{M}+T+1)\Gamma(\frac{\alpha}{M}+c_m^{01})\Gamma(c_m^{00}+1)\Gamma(\gamma+\delta)\Gamma(\delta+c_m^{10})\Gamma(\gamma+c_m^{11})}{\Gamma(\frac{\alpha}{M})\Gamma(T+1)\Gamma(\frac{\alpha}{M}+c_m^{00}+c_m^{01}+1)\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+\delta+c_m^{10}+c_m^{11})} \tag{5}$$

$$= \left(\frac{T!}{\prod_{t=1}^{T}(t+\frac{\alpha}{M})}\right)^{M} \left(\frac{\alpha}{M}\right)^{M_+} \prod_{m=1}^{M_+} \frac{\Gamma(\frac{\alpha}{M}+T+1)\prod_{i=1}^{c_m^{01}-1}(i+\frac{\alpha}{M})c_m^{00}!\Gamma(\gamma+\delta)\Gamma(\delta+c_m^{10})\Gamma(\gamma+c_m^{11})}{T!\Gamma(\frac{\alpha}{M}+c_m^{00}+c_m^{01}+1)\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+\delta+c_m^{10}+c_m^{11})} \tag{6}$$

Now we put equation (6) back into equation (2) and use the following results to compute the final limit:

$$
\begin{aligned}
\lim_{M\to\infty} \frac{M!}{M_0! M^{M_+}} &= \lim_{M\to\infty} \frac{\prod_{m=1}^{M_+}(M-m+1)}{M^{M_+}} \\
&= \lim_{M\to\infty} \left(\frac{M^{M_+} - \frac{(M_+-1)M_+}{2} M^{M_+-1} + \cdots + (-1)^{M_+-1}(M_+-1)!M}{M^{M_+}} \right) \\
&= \lim_{M\to\infty} \left(1 - O\left(\frac{c}{M}\right)\right) = 1, \qquad \text{for some constant } c \\
\lim_{M\to\infty} \prod_{i=1}^{c_m^{01}-1} (i + \frac{\alpha}{M}) &= \lim_{M\to\infty} \left((c_m^{01}-1)! + O\left(\frac{c}{M}\right)\right) = (c_m^{01}-1)!, \qquad \text{for some constant } c \\
\lim_{M\to\infty} \left(\frac{T!}{\prod_{t=1}^{T}(t+\frac{\alpha}{M})} \right)^M &= \lim_{M\to\infty} \prod_{t=1}^{T} \left(\frac{1}{1+\frac{\alpha\frac{1}{t}}{M}} \right)^M \\
&= \exp\left(-\alpha \sum_{t=1}^{T} \frac{1}{t}\right) \\
&= \exp(-\alpha H_T),
\end{aligned}
$$

where H_T is the T'th harmonic number. Putting everything together, we find the following limit

$$
\lim_{M\to\infty} p([\boldsymbol{S}]) = \frac{\alpha^{M_+}}{\prod_{h=1}^{2^T-1} M_h!} \exp\{-\alpha H_T\} \prod_{m=1}^{M_+} \frac{(c_m^{01}-1)! c_m^{00}! \Gamma(\gamma+\delta)\Gamma(\delta+c_m^{10})\Gamma(\gamma+c_m^{11})}{(c_m^{00}+c_m^{01})!\Gamma(\gamma)\Gamma(\delta)\Gamma(\gamma+\delta+c_m^{10}+c_m^{11})}. \qquad (7)
$$