

# AC Power: A Worked Example

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The voltage in a circuit is  $\tilde{V} = 240\angle 0$  and the impedance of the circuit at 50Hz is  $\tilde{Z} = 48 + j36$ .

In the following we work through the analysis of this setup, tackling many of the calculations you will be required to make in the Cambridge 1A course. A notational note:  $\tilde{V}$ ,  $\tilde{I}$ , and  $\tilde{Z}$ , are used to represent complex quantities with a magnitude and phase;  $V$ ,  $I$ , and  $Z$  are used to represent their polar magnitudes.

## Circuit setup

What does this circuit look like? The truth is we don't know. It could be made up of many different components which combine to give this impedance. The imaginary component of the impedance (called the *reactance*) is positive and this implies that the circuit is overall *inductive* rather than capacitive. This means that we can model the circuit using just a resistor and an inductor. But in which configuration should we place the resistor and inductor: in series or parallel (see figure 1)? It doesn't actually matter - there exists both a series combination and a parallel combination of a resistor and an inductor which has this impedance. However, as the impedance is given in rectangular form (aka Cartesian form) it is significantly easier to work with the series representation: we recall that the impedance of a resistor is series with an inductor is given by,

$$\tilde{Z} = R^{(s)} + j\omega L^{(s)} \quad (1)$$

To find the value of  $R$  and  $L$  we equate real and imaginary parts,

$$\begin{aligned} R^{(s)} &= 48\Omega \\ \omega L &= 36 \Rightarrow L^{(s)} = 0.11\text{H} \end{aligned} \quad (2)$$

remember  $\omega = 2\pi f = 100\pi$  as we are told the frequency is 50Hz.

To find the equivalent parallel representation we use the equation for the impedance of a resistor in parallel with an inductor,

$$\tilde{Z} = \frac{jR^{(p)}\omega L^{(p)}}{R^{(p)} + j\omega L^{(p)}} \quad (3)$$

Equating real and imaginary parts with this equation requires several lines of maths to deduce that,  $R^{(p)} = 75\Omega$  and  $L^{(p)} = 0.32\text{H}$ . For all the subsequent analysis we could use either configuration but to save repetition we will

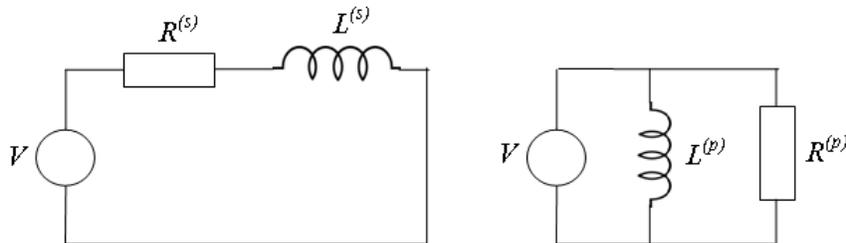


Figure 1: Series and parallel configurations of a resistor and inductor. Either of these circuits could be used to model an impedance of  $48 + j36$

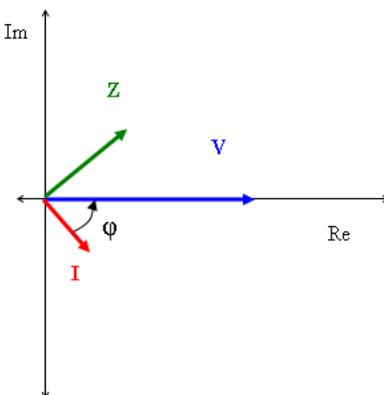


Figure 2: Argand diagram showing the complex voltage, impedance, and current.

restrict ourselves to just considering the series setup. Nearly all of the quantities that are calculated depend purely on the overall impedance and so are invariant to the series or parallel setup. The only differences are in calculating individual voltages and currents. A good test of your understanding would be to perform the calculations for the parallel configuration and ensure you get the same results.

## Current, phase, and power factor

To find the current in the series circuit we use Ohm's Law as usual,

$$\tilde{I} = \frac{\tilde{V}}{\tilde{Z}} = \frac{240\angle 0}{48 + j36} = \frac{240\angle 0}{60\angle 39.9^\circ} = 4\angle -36.9^\circ A \quad (4)$$

The voltage, impedance, and current are shown on the Argand diagram in figure 2. Note that since,

$$\arg\left(\frac{\tilde{V}}{\tilde{Z}}\right) = \arg(\tilde{V}) - \arg(\tilde{Z}) \quad (5)$$

impedances with positive imaginary components will lead to currents which *lag* the voltage.

The **phase angle** is given by,

$$\phi = \arg(\tilde{V}) - \arg(\tilde{I}) = 36.9^\circ \quad (6)$$

i.e. the angle from the current to the voltage. Therefore, positive phase angles mean that the current lags the voltage, and thus are called *lagging*, and negative phase angles mean that the current leads the voltage, and are called *leading*.

The **power factor** is defined as,

$$\text{power factor} = \frac{P}{S} = \cos(\phi) \quad (7)$$

As this is always a positive number the tag 'leading' or 'lagging' is usually added to describe the phase difference. The circuit in our example therefore has a power factor of  $\cos(36.9) = 0.8$  lagging.

## Real, reactive, and apparent power

**Real power**,  $P$ , is the amount of power consumed by the resistive components. It can be calculated from,

$$P = VI \cos(\phi) = \begin{cases} \frac{V^2}{R} & \text{for parallel components} \\ I^2 R & \text{for series components} \end{cases} \quad (8)$$

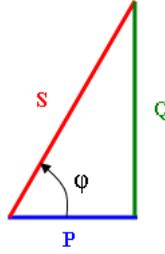


Figure 3: The power triangle

where  $R$  is the total resistance of the circuit and we are using the polar magnitudes. Conservation of real power means that,  $P_T = P_1 + P_2 + \dots$ . In our example  $P$  is 768W.

**Reactive power**,  $Q$ , is the amount of power consumed or supplied by the reactive components, i.e. capacitors and inductors. It can be calculated from,

$$Q = VI \sin(\phi) = \begin{cases} \frac{V^2}{X} & \text{for parallel components} \\ I^2 X & \text{for series components} \end{cases} \quad (9)$$

where  $X$  is the total reactance of the circuit. N.B. be careful of the signs as  $X$  and  $Q$  can be positive or negative. The sign of the reactive power should be the same as that of the circuit reactance. Therefore, lagging power factors are caused by positive reactive power and leading power factors by negative reactive power. Note that capacitors supply reactive power ( $Q$  negative) and inductors consume it ( $Q$  positive). Conservation of reactive power means that,  $Q_T = Q_1 + Q_2 + \dots$ . In our example  $Q$  is 576VAR.

**Apparent power** is the magnitude of the total power, volts times amps, supplied by the source. It is given by,

$$S = \sqrt{P^2 + Q^2} = V_{rms} I_{rms} \quad (10)$$

N.B. apparent power is **not** additive in the same way that real and apparent power are. That is, in general,  $S_T \neq S_1 + S_2$ . In our example  $S$  is 960VA.

Figure 3 shows the **power triangle**. This relates the *real*, *reactive*, and *apparent* powers. The phase angle can also be seen, confirming the relationship between the power factor and the ratio of  $P$  and  $S$ .

**Why is reactive power bad?** In an AC circuit voltage and current vary sinusoidally. This means that current flows backwards and forwards between the load and the source. If the load is purely resistive, the voltage and the current are in phase with each other. This means that their product, the power delivered to the load, is always greater or equal to zero, i.e. power always flows into the load. However, capacitive or inductive components in the load store energy and cause the current to move out of phase with the voltage. This results in a part of the AC cycle where the product of voltage and current is negative, i.e. power is flowing from the load back into the source, see figure 4. This exchange of energy between the capacitors/inductors and the source causes a current to flow which does not contribute to powering the load but which does cause power losses in the transmission system.

**Power factor correction** uses capacitors and inductors to bring the power factor closer to unity. This reduces the wasted reactive power. Energy will pass back and forth between the capacitive/inductive parts of the load and the corrective components rather than the source. This means that extra current is not drawn in the transmission lines.

Let's say that we want to improve our power factor of 0.8 lagging to 0.9 lagging. To calculate the required component first we deduce the amount of reactive power that needs to be supplied or consumed:

$$Q_{0.9} = P \tan(\arccos(0.9)) = \pm 371.96 \text{VAR} \quad (11)$$

The power factor is lagging and therefore we want the positive answer. Note that we **cannot** use  $VI \sin(\phi)$  straight off here as the current will have changed in the new circuit. Comparing the new  $Q$  to the reactive power we currently have, 576VAR, we see we need to generate 204.04VAR (generate because  $371.96 - 576$  is negative).

Using series corrective components is problematic as the current will change meaning that the total real power will also change. It is much easier to place the corrective component in parallel with the load.

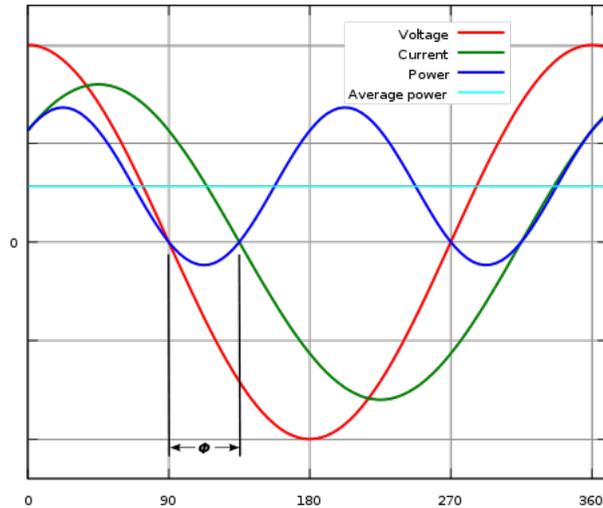


Figure 4: The problem caused by reactive loads: the phase difference introduced by reactive components means that for part of the AC cycle electrical energy is flowing from the load, where it has been stored in the reactive components, back to the source. This causes losses in the transmission system.

Remember that capacitors generate VARS and, as we are placing the capacitor in parallel, we can use equation 9,

$$Q_C = \frac{V^2}{X_C} = -\omega CV^2 = -204.04 \quad (12)$$

which implies that the capacitor should be  $11.3\mu\text{F}$  if the frequency is 50Hz (remember that  $\omega = 2\pi f$ ).

## Sign summary

In the following I have assumed  $\arg(V) = 0$ :

$$+ve X \Rightarrow +ve \phi \Rightarrow \text{lagging p.f.} \Rightarrow +ve Q$$

$$-ve X \Rightarrow -ve \phi \Rightarrow \text{leading p.f.} \Rightarrow -ve Q$$