

# RLC Resonant Circuits

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## 1 Capacitors and Inductors

There is a lot of inconsistency when it comes to dealing with reactances of complex components. The format followed in this document is as follows. The **impedance**,  $Z$ , of a component or a circuit is defined as,

$$Z = R + jX \quad (1)$$

where  $R$  is the resistance,  $j$  is the imaginary unit, and  $X$  is the reactance. Note that the imaginary unit is *outside* the reactance and there is a plus sign between  $R$  and  $jX$ .

Capacitors and Inductors are both components which can store energy: capacitors store it in an electric field and inductors in a magnetic field. Ideal capacitors and inductors are assumed to have zero resistance and so have a purely imaginary impedance,

$$\begin{aligned} Z_C &= \frac{1}{j\omega C} = -\frac{j}{\omega C} \\ Z_L &= j\omega L \end{aligned} \quad (2)$$

Following the convention in equation 1 we define the reactances to be,

$$\begin{aligned} X_C &= -\frac{1}{\omega C} \\ X_L &= \omega L \end{aligned} \quad (3)$$

note that the minus sign is included *inside*  $X_C$ , this is to ensure we get a plus sign in equation 1. You will often see capacitive reactance being quoted without this minus sign - be very careful!

Figure 1 shows how the absolute value of the reactances of the inductor and the capacitor change with frequency. Note that the capacitive reactance, and hence impedance, heads towards infinity for DC currents and zero for high

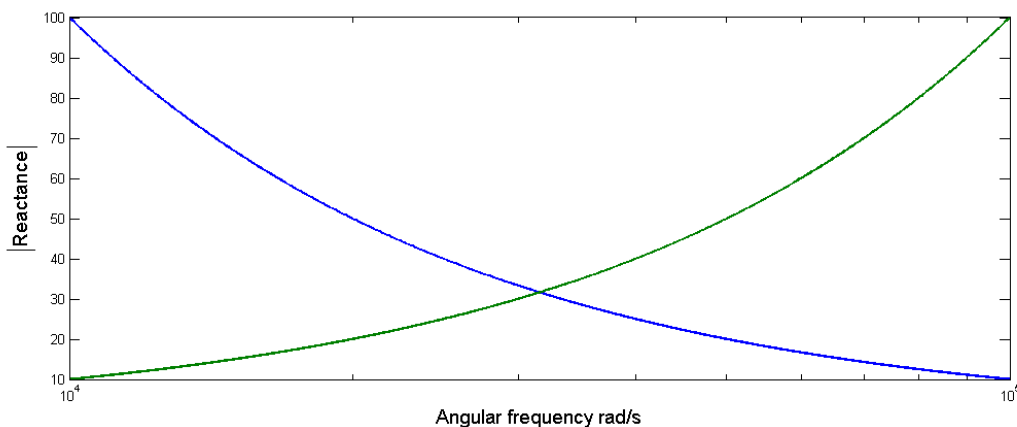


Figure 1: Reactance of a capacitor and a inductor w.r.t. angular frequency

frequency AC currents. This is what we expect - ideal capacitors are considered open circuits for DC current and short circuits for high enough frequency AC signals (this is why we can ignore some capacitors in transistor circuits at ‘mid-band’ frequencies).

## 2 Series Circuit

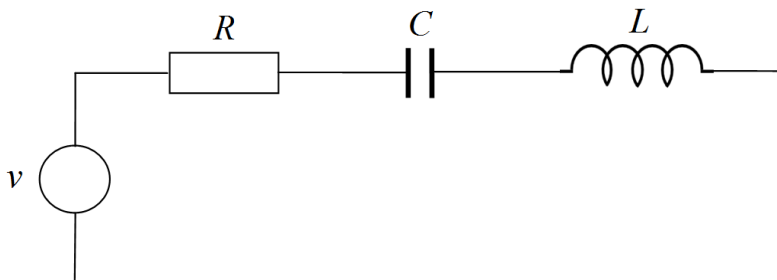


Figure 2: Series RLC circuit

Figure 2 shows a series resonant RLC circuit. The total impedance in the circuit is given by,

$$Z_{\text{total}} = R + Z_L + Z_C$$

$$= R + j(X_L + X_C) = R + j\left(\omega L - \frac{1}{\omega C}\right) \tag{4}$$

$$\tag{5}$$

Figure 3 shows how the total impedance changes with frequency. Figure 1 shows that the absolute reactances of

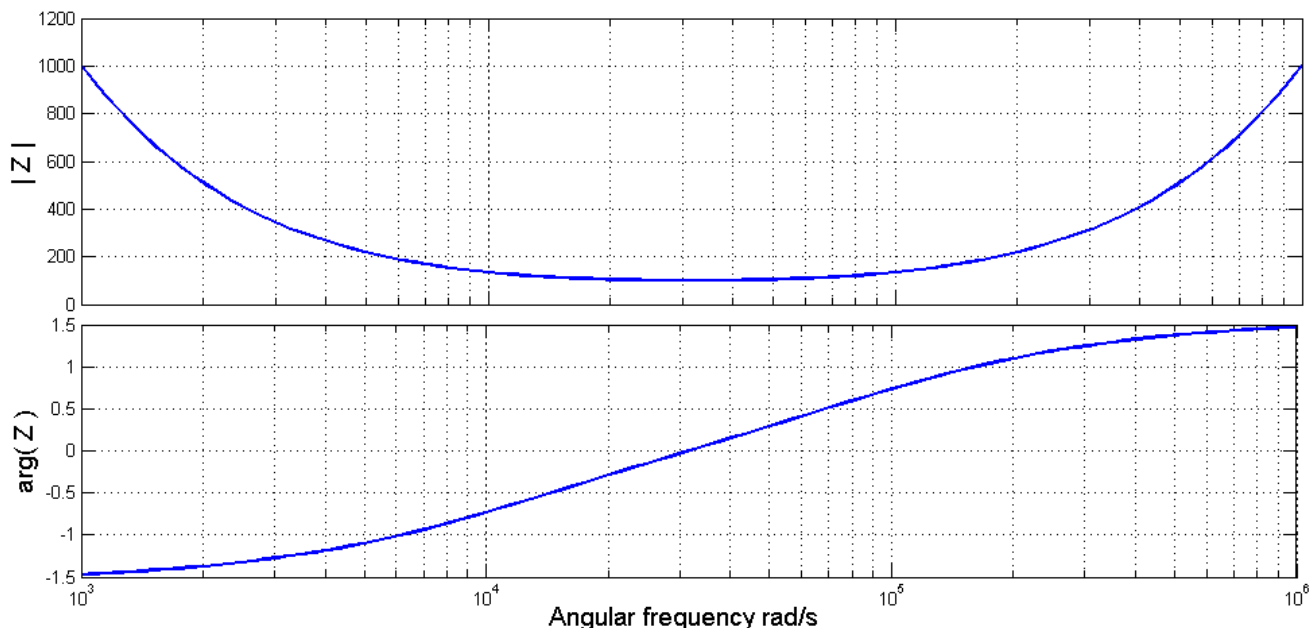


Figure 3: Magnitude and phase of impedance in a series RLC circuit

a capacitor and inductor will approach each other and cross as frequency increases. If we now recall that  $X_C$  is actually negative then it is clear that  $X_L + X_C$  must head towards zero in this frequency range. At the specific frequency where  $X_L = -X_C$ , the imaginary components of the impedance exactly cancel each other out (equation 4 shows this most clearly). At this frequency the impedance of the circuit has its smallest magnitude and a phase

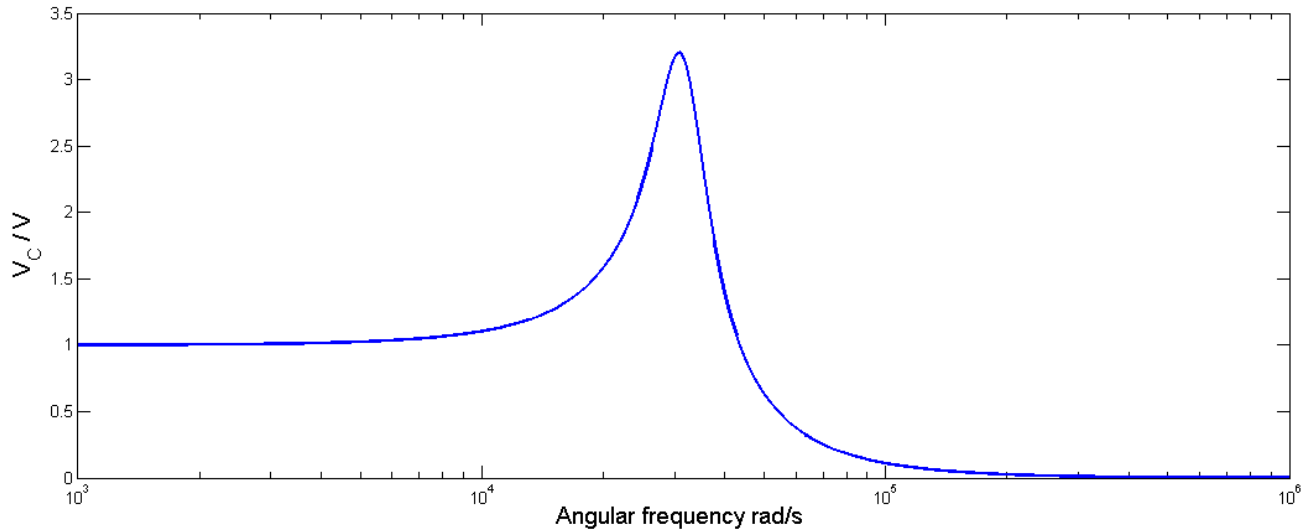


Figure 4: Voltage amplification over the capacitor in a series RLC circuit

angle of zero (remember only imaginary components can move phase angles away from zero). This is called the *resonant frequency* of the circuit,  $f_0$ . It is always the frequency which causes  $X_L$  and  $X_C$  to cancel each other out,

$$\begin{aligned}
 X_L + X_C &= 0 \\
 \Rightarrow X_L &= -X_C \Rightarrow \omega_0 L &= \frac{1}{\omega_0 C} \\
 \Rightarrow \omega_0^2 &= \frac{1}{LC} \\
 \Rightarrow \omega_0 &= \sqrt{\frac{1}{LC}}
 \end{aligned} \tag{6}$$

where  $\omega_0 = 2\pi f_0$ , the resonant angular frequency.

**You must be able to calculate the resonant frequency for arbitrary RLC circuits.**

To see the resonance effect consider the ratio of the voltage across the reactive components to the input voltage. For example, for the capacitor,

$$\frac{v_C}{v} = \frac{1}{1 - \omega^2 LC + j\omega RC} \tag{7}$$

This ratio is plotted against angular frequency for particular values of  $R$ ,  $L$ , and  $C$  in figure 4. The figure shows that the circuit exhibits voltage amplification properties. At the resonant frequency,

$$\begin{aligned}
 \frac{v_C}{v} &= \frac{1}{j\omega_0 RC} \\
 \frac{v_L}{v} &= \frac{j\omega_0 L}{R}
 \end{aligned} \tag{8}$$

It is important to note that as this is a passive circuit the total amount of power dissipated is constant.

### 3 Parallel Circuit

Figure 5 shows a parallel resonant RLC circuit. It is the ‘dual’ of the series circuit in that the voltage and current exchange roles. This means that the circuit has a current gain rather than a voltage gain and that the impedance is

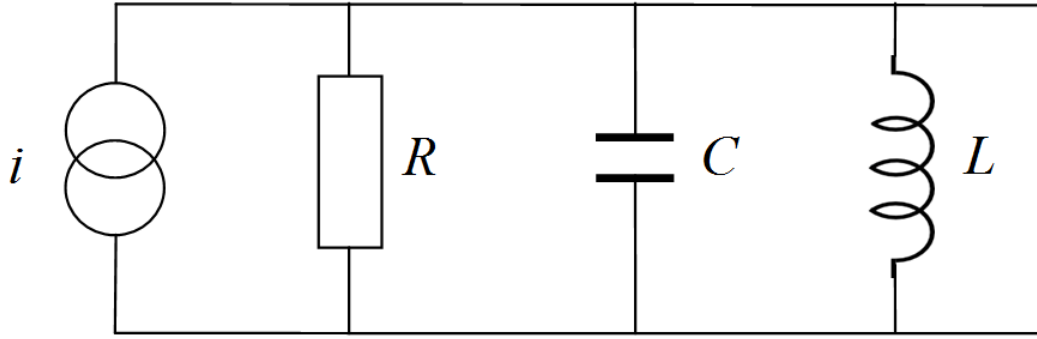


Figure 5: Parallel RLC circuit

maximised at the resonant frequency rather than minimised. The total impedance in the circuit is given by,

$$Z_{\text{total}} = R \parallel Z_L \parallel Z_C = \frac{R}{1 - jR(\frac{1}{X_C} + \frac{1}{X_L})} = \frac{R}{1 + jR(\omega C - \frac{1}{\omega L})} \quad (9)$$

$$(10)$$

Figure 6 shows the magnitude and phase of the impedance of the circuit. Once again the resonant peak comes when  $X_C = -X_L$  and hence the resonant frequency is given by the same relationship,

$$\omega_0 = \sqrt{\frac{1}{LC}} \quad (11)$$

The current gain can be calculated by looking at the current in each of the arms. Take the capacitor again,

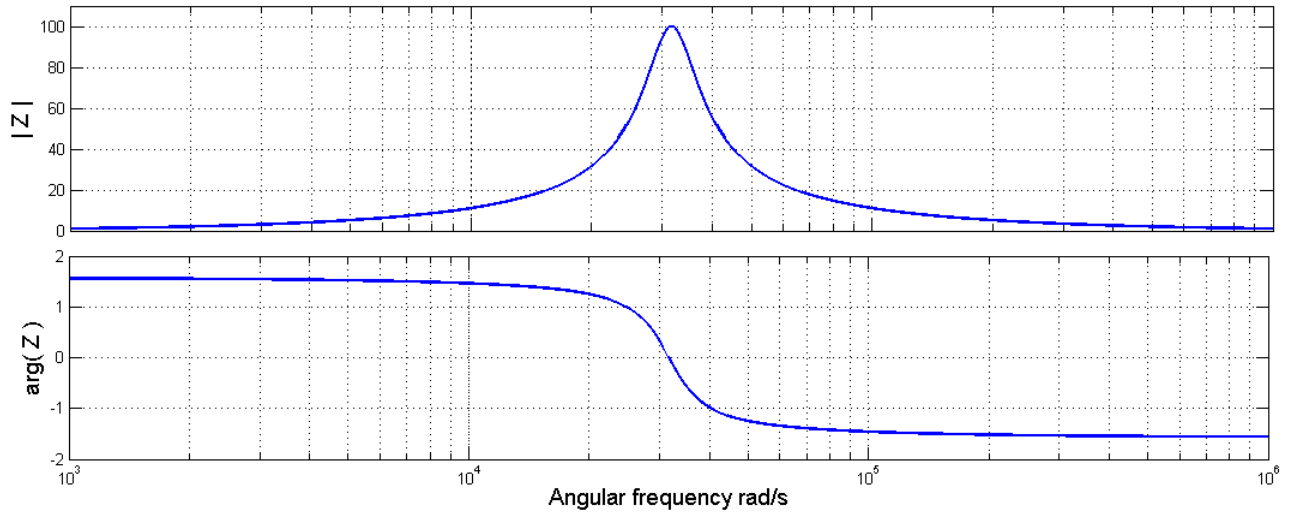


Figure 6: Magnitude and phase of impedance in a parallel RLC circuit

$$\frac{i_C}{i} = \frac{j\omega RC}{1 + jR(\omega C - \frac{1}{\omega L})} \quad (12)$$

The current gain magnitude is shown in figure 7. At the resonant frequency,

$$\frac{i_C}{i} = j\omega RC \quad (13)$$

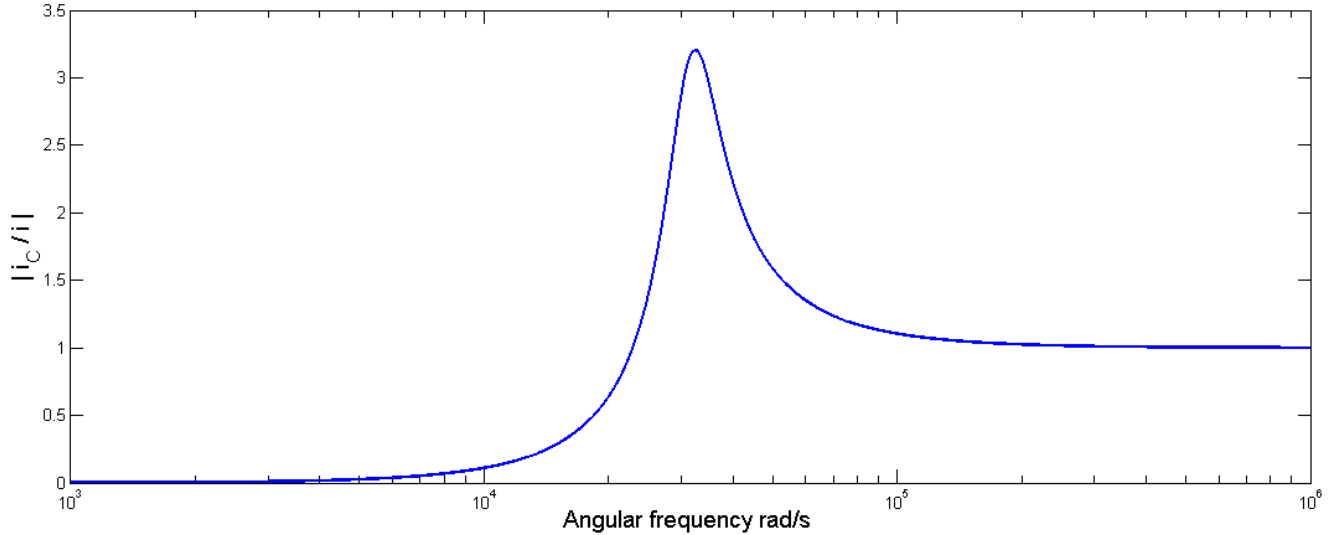


Figure 7: Current gain in the capacitor

## 4 Q factor

The Quality Factor, or Q Factor, describes how under-damped a resonator is; the higher the Q factor the less damping there is. In an electrical resonator circuit damping is caused by the loss of energy in resistive components. The Q factor can therefore be described as:

$$Q(\omega) = \omega \frac{\text{Maximum Energy Stored}}{\text{Power Loss}} \quad (14)$$

The Q factor depends on frequency but it is most often quoted for the resonant frequency  $\omega_0$ .

The maximum energy stored in a resonant circuit at the resonant frequency can be calculated from either the max energy stored in the capacitor or the max energy stored in the inductor - it is not the sum of these as when one is holding its max energy the other has zero energy in it. The relevant equations are:

$$\text{Max energy stored} = L I_{Lrms}^2 = C V_{Crms}^2 \quad (15)$$

Note that  $I_{Lrms}$  is the rms current through the inductor. For a simple series RLC circuit such as figure 2, this is just equal to the total rms current flowing in the circuit, however for a parallel RLC circuit this will *not* be the same. Similarly,  $V_{Crms}$  is the rms voltage across the capacitor. For the simple parallel RLC circuit shown in figure 5 this is just equal to the rms supply voltage but for the series RLC circuit it is given by a potential divider rule. Therefore, for series circuits it is in general simpler to calculate the max energy stored by considering the inductor and in parallel circuits by considering the capacitor.

Real power is only dissipated in the resistors,

$$P = V_{Rrms} I_{Rrms} = I_{Rrms}^2 R = \frac{V_{Rrms}^2}{R} \quad (16)$$

For the series RLC circuit it is easiest to consider currents,

$$Q_{\omega_0}^{(s)} = \omega_0 \frac{I_{rms}^2 L}{I_{rms}^2 R} = \frac{\omega_0 L}{R} \quad (17)$$

For the parallel circuit it is simpler to consider voltages,

$$Q_{\omega_0}^{(p)} = \omega_0 \frac{RC V_{Crms}^2}{V_{Crms}^2} = \omega_0 CR \quad (18)$$

For high values of  $Q$  it is worth noting that the following definition is also mathematically accurate,

$$Q = \frac{\omega_0}{\Delta\omega} \quad (19)$$

where  $\Delta\omega$  is the circuit bandwidth, the distance between the two cut-off angular frequencies (aka the -3dB points).

## 5 Lossy Components

So far we have assumed that the inductor and the capacitor are ideal components with no internal resistances. In this section we look at the effect of a more realistic model of the reactive components on the resonant circuit.

### 5.1 Inductors

An inductor is essentially just a coil of wire and so often has a non-negligible resistance. This is most readily modelled as a resistor in series with the inductor as shown on the left in figure 8. The smaller this resistance is the more ideal the inductor is.

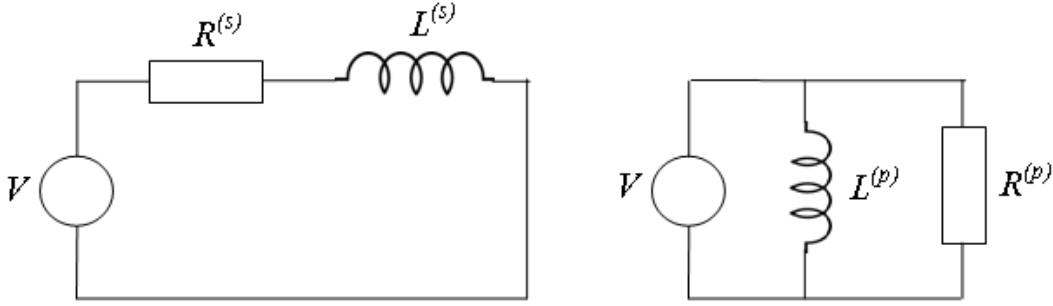


Figure 8: Series and parallel internal resistance for an inductor

We can define a  $Q$  factor for just the inductor and its internal resistance in the same way as we did for the whole circuit in the previous section. In fact the derivations are the same meaning,

$$Q_L^{(s)} = \frac{X_L^{(s)}}{R^{(s)}} \quad (20)$$

We could also define an equivalent circuit with a parallel resistance, as in the circuit on the right of figure 8. We can calculate the  $Q$  factor for this circuit using the same method, noting that,

$$\begin{aligned} I_{Lrms}^{(p)} &= \frac{V_{rms}}{\omega L^{(p)}} \\ I_{Rrms}^{(p)} &= \frac{V_{rms}}{R^{(p)}} \end{aligned} \quad (21)$$

which leads to,

$$Q_L^{(p)} = \frac{R^{(p)}}{X_L^{(p)}} \quad (22)$$

For the two circuits in figure 8 to be equivalent their impedances must be equal. This also implies that their  $Q$  factors must also be equal,

$$Q_L^{(s)} = Q_L^{(p)} \quad (23)$$

$$\frac{X_L^{(s)}}{R^{(s)}} = \frac{R^{(p)}}{X_L^{(p)}} \quad (24)$$

$$\triangleq Q_L \quad (25)$$

If we equate imaginary parts of the impedances of the two circuits and substitute in  $R_L^{(p)} = Q_L X_L^{(p)}$ , we get

$$X_L^{(p)} = X_L^{(s)} (1 + Q_L^{-2}) \quad (26)$$

Thus for large Q we can effect an approximation,

$$X_L^{(p)} \approx X_L^{(s)} \quad (27)$$

$$\frac{R_L^{(p)}}{X_L} \approx \frac{X_L}{R_L^{(s)}} \quad (28)$$

This relationship is often used in the course to allow you to switch between series

## 5.2 Capacitors

In an ideal capacitor the two plates are perfectly insulated from each other. However, in a real capacitor the insulation comes from the dielectric between the two plates which has a finite resistance. This allows a small current to flow between the two plates. A non-ideal capacitor is therefore modelled as an ideal capacitor in parallel with a resistor, similar to the setup in the right circuit of figure 8. The larger this resistor is the closer to ideal the capacitor is.

As with an inductor we can define a Q factor for the capacitor,

$$Q_C = \frac{R^{(p)}}{X_C^{(p)}} \quad (29)$$

The analysis is analogous to the inductor, thus for large Q we can write,

$$\frac{R_C^{(p)}}{X_C} \approx \frac{X_C}{R_C^{(s)}} \quad (30)$$