Charged Planes

Andrew McHutchon

Michaelmas 2012

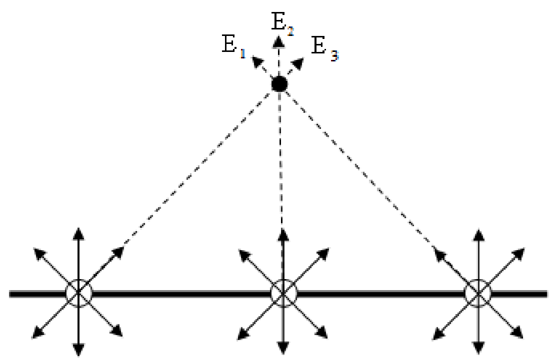

Figure 1: A charged infinite plane can be considered to be made up of an infinite number of point charges. The electric field at a point can be found by superposition of the electric fields coming from each of the point charges.

Imagine we have an infinite plane, holding a positive charge of Q on it. What is the electric field surrounding this plane? To calculate this we can consider the plane to be made up of an infinite collection of point charges, as shown in figure 1. Each of these point charges has a charge equal to the charge density σ and induces a radial electric field of magnitude,

$$E_i = \frac{\sigma}{4\,\pi\,\epsilon\,r^2} \tag{1}$$

Electric fields are linear, which means they obey the principle of superposition. This means that to find the total electric field at some point above the plane we can add up the electric field from all of the point charges. Electric fields are vector fields and so this addition must be a vector addition. We can see from figure 1 that, because of symmetry, each of the contributions from one side of the point we are interested in will be balanced by the point charges on the other side. This means that the resulting electric field must be perpendicular to the plane. We can make this argument for all points in the plane if the plane is infinite.

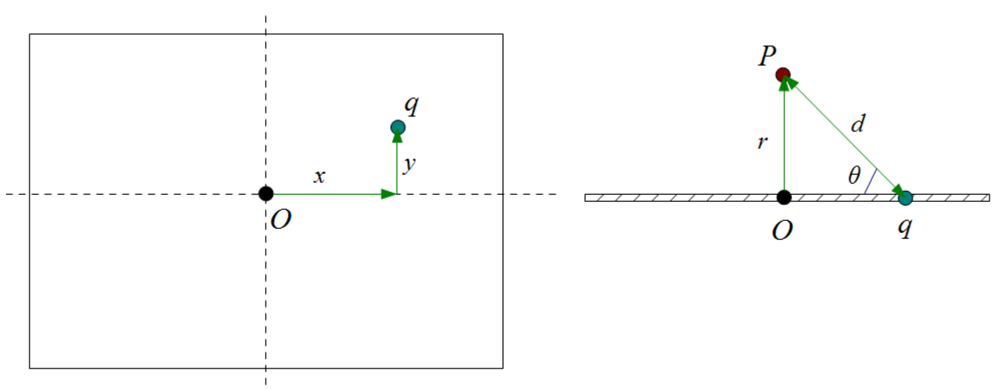

Figure 2: The setup to find the total electric field at a point P above an infinite charged plane via integration. Here we choose a Cartesian setup, a polar co-ordinate setup can also be used. We will sum all contributions from each infinitesimal point-charge q, which has a charge equal to the charge density σ.

In order to find the magnitude of the electric field we can either integrate, or much more simply use Gauss's law. The integration approach is as follows: define a point q to lie in the plane at a distance x along the x-axis and y along the y-axis, as shown in figure 2. Let's say we want to find the electric field at a point P, which is at a distance of r above the origin of the plane. We know from the above analysis that only the vertical component of the electric field from point q will contribute to the resultant electric field at P. To find the magnitude of E_P we therefore need to integrate,

$$E_P = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\sigma}{4\,\pi\,\epsilon\,d^2} \sin\theta \; dx \; dy \tag{2}$$

where d is the distance from q to P and θ is the angle between the plane and the vector from q to P. From the geometry we can write,

$$d = \sqrt{x^2 + y^2 + r^2} \tag{3}$$

$$\sin\theta = \frac{r}{\sqrt{x^2 + y^2 + r^2}} \tag{4}$$

Thus the integral is now,

$$\begin{aligned} E_P &= \frac{\sigma\,r}{4\,\pi\,\epsilon} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(x^2 + y^2 + r^2)^{3/2}} \; dx \; dy \\ &= \frac{\sigma\,r}{4\,\pi\,\epsilon} \int_{-\infty}^{\infty} \frac{2}{(y^2 + r^2)^{3/2}} \; dy \\ &= \frac{\sigma\,r}{4\,\pi\,\epsilon} \, \frac{2\,\pi}{r} \\ &= \frac{\sigma}{2\,\epsilon} \end{aligned} \tag{5}$$

Using Gauss's Law, we define a Gaussian surface, S, to be a rectangular box, with sides at a distance h above and below the plane and a cross-sectional area of A, as in figure 3.

$$Q = \oint_S \boldsymbol{D} \cdot d\boldsymbol{A} \tag{6}$$

Electric field lines are perpendicular to the plane and thus to the top and bottom of our Gaussian surface. The field lines are directed downwards through the bottom of the box and upwards through the top of the box, in other words they always point out of the box, $\boldsymbol{D} \parallel d\boldsymbol{A}$. No field lines pass through the sides of the box, hence the total area through which flux passes is equal to $2A$: the area of the top and bottom of the box. The total charge enclosed is equal to the charge density times the area of the plane inside the Gaussian surface, thus,

$$\begin{aligned} \sigma\,A &= \oint_S D \; dA \\ &= 2\,D\,A \\ \Rightarrow E &= \frac{\sigma}{2\,\epsilon} \end{aligned} \tag{7}$$

We have shown that the electric field above an infinite plane is uniform and perpendicular to the plane. We will now consider the case where we have two such planes, the second of which has a total charge of $-Q$. This setup is shown in figure 4.

To calculate the electric field at some point in this system we once again can use superposition: adding together the electric fields from both plates. Considering figure 4 we can see that the only location where the field lines from the two planes are in the same direction is inbetween the two planes. In every other location the field lines will oppose each other. As the electric field from both planes is uniform the distance from the plane does not matter. Thus inbetween the planes the electric fields will add and everywhere else they will interfere with each other. We defined the two planes to have equal but opposite charges and so the electric field outside the planes will be exactly zero.

$$E = \begin{cases} \dfrac{\sigma}{\epsilon} & \text{between the planes} \\ 0 & \text{elsewhere} \end{cases} \tag{8}$$

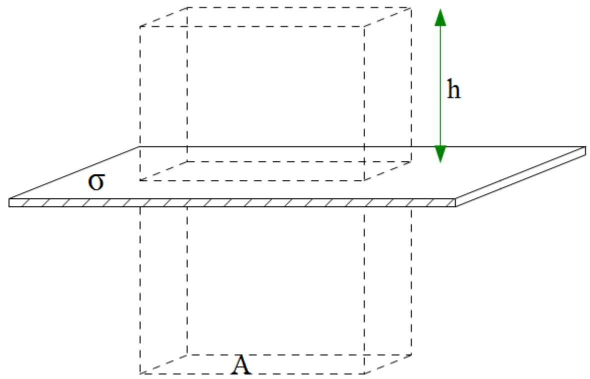

Figure 3: The setup for application of Gauss's Law. The Gaussian surface is a rectangular box of height $2h$ and cross-sectional area A, which is placed across as the charged plane. The plane has charge density σ and so the surface encloses a total charge of σA. The electric field lines are perpendicular to the plane, and so the total electric flux through the Gaussian surface is $2DA$.

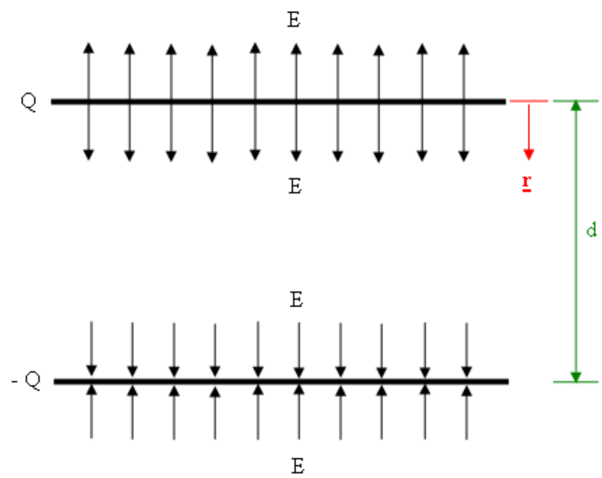

Figure 4: Two charged infinite planes with equal but opposite charges. The electric field from each plane is uniform and can be combined via superposition. The fields will oppose each other everywhere apart from between the planes where they combine. This results in an electric field of $2\,E$ between the planes and zero everywhere else.

Of course this is only true if the two planes are infinite. However, in practice it is a good approximation so long as the distance between the planes is small compared to their area. In the Cambridge Part 1A course, unless specifically told otherwise, you can always make this assumption. This means that for two parallel charged plates of equal but opposite charge the electric field can be assumed to be uniform between the plates and zero everywhere else.