Electromagnetism Laws and Equations

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Contents

1 Electrostatics

1.1 Electric $\boldsymbol{E}$- and $\boldsymbol{D}$-fields

Electric fields are linear (obey superposition), vector fields. $\boldsymbol{E}$ is the electric field strength and $\boldsymbol{D}$ is known as the electric flux density. They are related by:

$$\boldsymbol{D} = \epsilon_0 \, \epsilon_r \, \boldsymbol{E}$$

where, ϵ_0 is the *permittivity of free space*, and ϵ_r is the *relative permittivity*. The $\boldsymbol{D}$-field is independent of the material, whereas the $\boldsymbol{E}$-field is not. This is an important use of $\boldsymbol{D}$. The product $\epsilon_0 \, \epsilon_r$ is often written ϵ.

1.1.1 Electrostatic Force

The electric field strength and the force exerted on a charged particle by the field are related by,

$$\boldsymbol{F} = \boldsymbol{E}q \tag{1}$$

where q is the charge on the particle in Coulombs.

1.1.2 Uniform electric fields

A uniform field is one in which the electric field is the same at every point. It is most commonly encountered between two parallel, conducting plates, ignoring edge effects. The equation for the magnitude of the electric field in this setup is:

$$E = -\frac{V}{d} \tag{2}$$

where, V is the voltage difference, and d the distance, between the plates. The electric field flows from the positive plate to the negative, the opposite direction to the direction of increasing voltage; this results in the negative sign.

Common mistake: equation 2 is only valid for uniform electric fields, a common mistake is to use equation 2 for non-uniform fields.

1.2 Gauss's Law

Gauss's Law is a cornerstone of electrostatics. It is primarily used to relate electric field strength to charge.

Gauss's law states that: The net electric flux through any closed surface is proportional to the enclosed electric charge:

$$Q = \oint_S \boldsymbol{D} \cdot d\boldsymbol{A} \tag{3}$$

where, Q is the enclosed charge, $\boldsymbol{D}$ is the electric flux density at area element $d\boldsymbol{A}$ of closed surface S. $d\boldsymbol{A}$ is a vector and its direction is normal to plane of the area element, pointing to the 'outside' of the Gaussian surface. Gauss's law is true for *any* closed surface. However, if we can choose the surface S carefully the equation can be simplified significantly: we aim to choose a Guassian surface such that,

1. $\boldsymbol{D}$ and $d\boldsymbol{A}$ are parallel everywhere on S
2. the magnitude of $\boldsymbol{D}$ is constant everywhere on S

The first condition reduces the dot product to just the product of the magnitudes. The second condition allows the magnitude of $\boldsymbol{D}$ to be taken out of the integral, as it is not a function of the position on the surface. These steps allow the manipulation of equation 3 as follows,

$$Q = \oint_S \boldsymbol{D} \cdot d\boldsymbol{A} = D \oint_S dA = DA \tag{4}$$

where A is the total area of S. *N.B.: This simplification only holds if* $\boldsymbol{D} \parallel d\boldsymbol{A}$*!*

An example:

Using Gauss's Law, find the electric field strength $\boldsymbol{E}$ *at a distance* r *from a point charge* Q*.*

Define the Gaussian surface to be a sphere of radius r centred on the point charge, as shown in figure 1. For a particular value of r, $\boldsymbol{D}$ has constant magnitude and is perpendicular to the surface, therefore the integral becomes,

$$Q = D \oint_S dA = D \times 4\,\pi\,r^2$$

$$E = \frac{Q}{4\,\pi\,\epsilon\,r^2} \tag{5}$$

This is a derivation of Coulomb's Law. Gauss's Law is often written in terms of $\boldsymbol{E}$ rather than $\boldsymbol{D}$. The result is exactly the same (indeed we converted $\boldsymbol{D}$ to $\boldsymbol{E}$ in the above example).

Common mistake: consider the following question: find the electric field at a distance r from the centre of a uniformly charged sphere of radius r_0, where $r > r_0$. As with the point charge, the field lines are radial and so we want to use a sperical Gaussian surface. A common mistake is for students to set the radius of the Gaussian surface to be fixed at r_0. *The value of E calculated using Gauss's law is found on the Gaussian surface.* Therefore using a sphere of r_0 only calculates the electric field at the surface of the sphere and not a general location r as the question asked us to do. If you are asked to find the electric field at a distance r from the centre of the sphere you must use a Gaussian surface with a radius r not with radius r_0.

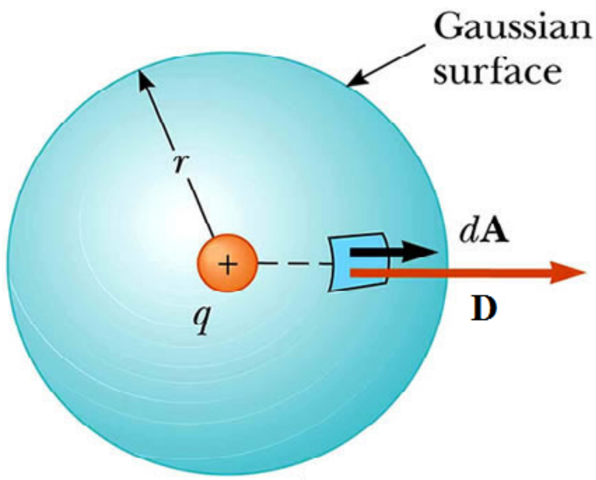

Figure 1: Gauss's Law applied to a point charge. The electric field lines are straight, radial lines and so we choose a spherical Gaussian surface, with a variable radius r.

1.3 Coulomb's Law

Coulomb's Law describes the electrostatic interaction between two charged particles. It can be derived by combining the equation for the electric field around a spherical charge, equation 5, with the equation for electric force, equation 1. It is an inverse-square law, and is given by:

$$\boldsymbol{F}_{21} \;=\; \frac{q_1\, q_2}{4\,\pi\,\epsilon\, r^2}\, \hat{\boldsymbol{r}}_{\boldsymbol{21}} \tag{6}$$

where, $\boldsymbol{F}_{21}$ is the force on particle 2 from particle 1, r is the distance between the particles, and $\hat{\boldsymbol{r}}_{\boldsymbol{21}}$ is a unit vector in the direction of particle 2 from particle 1. Note that when both particles have the same sign of charge then the force is in the same direction as the unit vector and particle is repelled. Take care to get the particles the correct way around - are you calculating the force on particle 1 or 2? Does the unit vector point in the right direction?

1.4 Electric Potential

The electric potential of a point is the amount of work that has to be done, per unit charge, to move a point charge from a place of zero potential to that point. This is the electric potential energy of the point divided by the charge at that point - or electric potential energy per unit charge. The difference in electric potential between two points is what we commonly refer to as 'voltage' - or *electric potential difference*. Both of these quantities have SI units of volts. An analogy to electric potential is gravitational potential - or gravitational potential energy per unit mass.

The equation for the electric potential of a point p is given by the line integral,

$$V(p) \;=\; -\int_C \boldsymbol{E} \,\cdot\, d\boldsymbol{l} \tag{7}$$

where C is an arbitrary curve which connects a point of zero potential to the point p, E is the electric field that is experienced by the curve element $d\boldsymbol{l}$. Note that the integral involves a dot product, which indicates that the electric potential is only changed when the curve moves with or against the electric field, rather than perpendicular to it. Also, **note the minus sign**! This comes about because electric potential is increased when the curve element is in the opposite direction to the electric field, that is, our imaginary charge is being moved *against* the electric field. This should be familiar: potential energy always increases with movement against a force.

Rather than evaluating the integral from a point of zero potential, more often it is useful to define the equation in terms of potential difference:

$$V(p_2) \;-\; V(p_1) \;=\; -\int_{p_1}^{p_2} \boldsymbol{E} \,\cdot\, d\boldsymbol{l} \tag{8}$$

where $V(p_2)$ and $V(p_1)$ are the electric potentials at p_2 and p_1 respectively, and the integral is evaluated along *any* curve joining the two points. Note which way around $V(p_2)$ and $V(p_1)$ are, the same order as the limits. *It is very easy to make a minus sign mistake here, always label clearly on a diagram where points p_1 and p_2 are.*

Equation 8 allows you to compute the potential difference between any two points in a electric field. The integral in equation 8 is over a dot product between the electric field and any path between points p_1 and p_2. For simple setups, such as the ones you will come across in Part 1A, we can often choose this path to be parallel to the electric field and hence replace the dot product with the product of magnitudes, in a similar way to Gauss's Law. However, we generally *cannot* take the electric field E outside the integral, like we did with Gauss's Law, as the electric field won't be constant between the two points, i.e. the magnitude of E is a function of l. E can only be taken out of the integral in equation 8 in uniform electric fields, i.e. between parallel plates.

Common mistake: to not treat E as a function of l and to take it out of the integral.

An example: Consider the setup shown in figure 2. We want to calculate the potential difference of p_2 relative to p_1 resulting from the electric field created by the point charge Q.

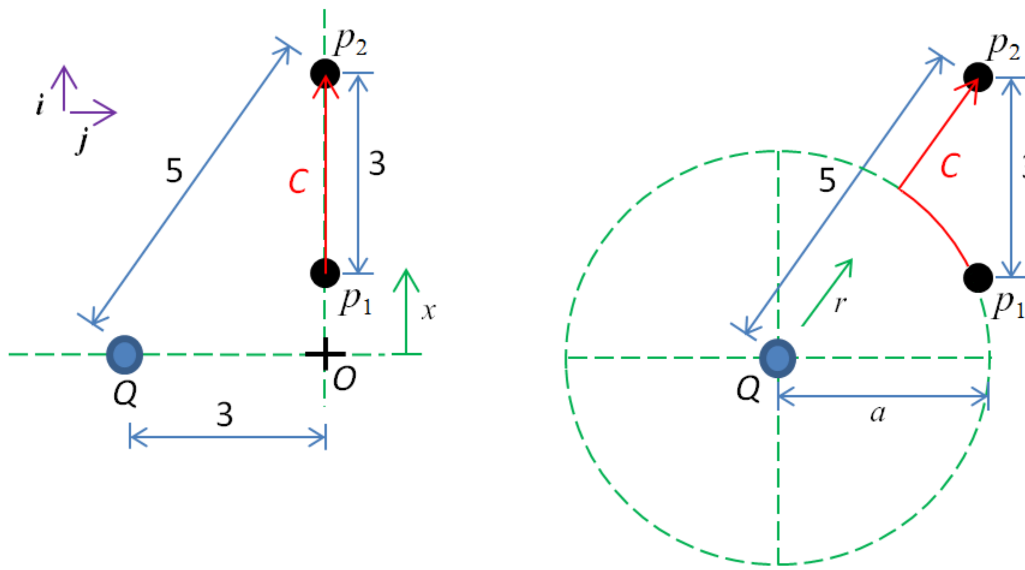

Figure 2: The setup for the potential difference worked example. The question requires us to find the potential difference between the points p_1 and p_2 in the presence of the electric field created by the point charge Q.

For our first solution we shall pick the path C to be a straight line from p_1 to p_2. We parameterise the path by the variable x, the origin of which we set to be at the point O. We also define unit vectors $\boldsymbol{i}$ which points along C and $\boldsymbol{j}$ in the perpendicular direction to C, as shown. This gives,

$$C = x\boldsymbol{i} \Rightarrow d\boldsymbol{l} = dx\boldsymbol{i} \tag{9}$$

We will integrate from p_1 at $x = 1$ to p_2 at $x = 4$. In order to solve the integral in equation 8 we need to write the electric field in terms of our variable of integration, x. By using Gauss's Law or Coulomb's law and figure 2 we can write,

$$\boldsymbol{E} = \frac{Q}{4\pi\epsilon r^3}\boldsymbol{r} \tag{10}$$

$$= \frac{Q}{4\pi\epsilon(x^2 + 3^2)^{3/2}}(x\boldsymbol{i} + 3\boldsymbol{j}) \tag{11}$$

where, $\boldsymbol{r} = r\hat{\boldsymbol{r}}$, which is why we divide by r^3. Combining with equation 8,

$$\begin{aligned}
V(p_2) - V(p_1) &= -\int_{p_1}^{p_2} \boldsymbol{E} \cdot d\boldsymbol{l} \\
&= -\frac{Q}{4\pi\epsilon} \int_1^4 \frac{1}{(x^2+3^2)^{3/2}} \,(x\boldsymbol{i} + 4\boldsymbol{j}) \cdot dx\boldsymbol{i} \\
&= -\frac{Q}{4\pi\epsilon} \int_1^4 \frac{x}{(x^2+3^2)^{3/2}} \,dx \\
&= \frac{Q}{4\pi\epsilon} \left[\frac{1}{\sqrt{x^2+3^2}}\right]_1^4 \\
&= \frac{Q}{4\pi\epsilon} \left[\frac{1}{5} - \frac{1}{\sqrt{10}}\right] \qquad (12)
\end{aligned}$$

We could have made this calculation easier by careful choice of the path from p_1 to p_2 (remember we can choose any path we like). The right hand plot in figure 2 shows an alternative path, which splits into an arc centred on the charged paticle and a radial line. This path looks like it would be more complex to use, however note that the curved path is always perpendicular to the field lines and the radial line is always parallel to the electric field. It is easy to see from the dot product in the voltage equation that the perpendicular part of the path will have no contribution to the voltage. We can use simple trigonometry to find that $a = \sqrt{10}$, hence,

$$\begin{aligned}
V(p_2) - V(p_1) &= -\frac{Q}{4\pi\epsilon} \int_{\sqrt{10}}^{5} \frac{1}{r^2} \,dr \\
&= \frac{Q}{4\pi\epsilon} \left[\frac{1}{r}\right]_{\sqrt{10}}^{5} \qquad (13)
\end{aligned}$$

which gives the same result.

1.4.1 Potential difference in a capacitor

In the Cambridge course, a common application of equation 8 is to find the voltage between the plates of a capacitor. To do this, we first use Gauss's Law to find the electric field at a variable distance r between the plates and then combine this with equation 8. When doing this it is always easiest to integrate in the direction of the electric field. For example, for the parallel plate capacitor shown in figure 3, we integrate from the top plate to the bottom plate,

$$V(r_2) - V(r_1) = -\int_{r_1}^{r_2} E(r)\,dr \qquad (14)$$

$$= -\frac{Q}{\epsilon A}(r_2 - r_1) \qquad (15)$$

where we used Gauss's law to find $E = Q/\epsilon A$. Note that for a parallel plate capacitor the electric field is uniform and hence the integration is trivial. However, for any other kind of capacitor, for example concentric spheres, *this will not be the case* - E will be a function of r, and hence must be integrated correctly.

1.5 Capacitors

1.5.1 Introduction to capacitors

Imagine connecting a conducting plate to the negative terminal of a voltage source and another plate to the positive terminal. With the two plates unconnected to each other (except via the source) the circuit is incomplete and basic electrical theory tells us that a steady-state current cannot flow. However, the voltage source will cause a *transient* current to flow as (positive) charge is moved from the plate on the negative side of the source to the plate on the positive side. A good analogy is that of a pump in a closed gas pipe - the pressure created by the pump forces gas

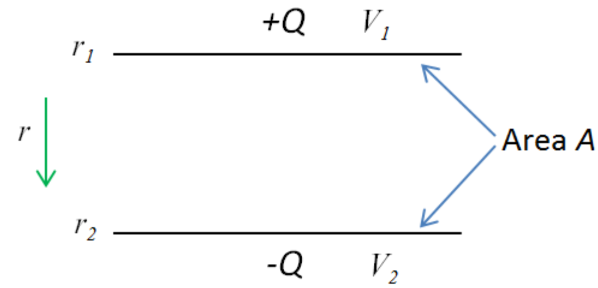

Figure 3: Capacitor made from two parallel plates, each of area A. Charge Q has been transferred from the bottom plate to the top plate via the connected voltage source (not shown). This leaves the top plate with a net charge of $+Q$ and the bottom plate with $-Q$. Positive charge is transferred from the negative terminal of a voltage source to the positive terminal, therefore as positive charge has been transferred from the bottom plate to the top plate $V_1 > V_2$.

through it into the closed end of the pipe. This flow of gas cannot be maintained indefinitely: as more gas is forced into the pipe the pressure inside the pipe increases until it matches the pressure created by the pump. At this point equilibrium is reached and the flow stops. The same phenomenon occurs in a capacitor - charge is transferred between the plates until the build up of electrostatic force matches the electromotive force created by the voltage source (note, I am being slightly liberal with the word 'force' here).

For the setup described in the previous paragraph the amount of charge that can be transferred before equilibrium occurs is very small. We can increase the amount of charge that can be stored by placing the negative and positive plates in close proximity to each other. The attractive electrostatic force between the plates combines with the voltage source meaning that more charge must build up on the plates before equilibrium is reached.

The amount of charge that can be transferred between the plates of a capacitor is related to the potential difference applied between the plates via the following equation,

$$Q = CV \tag{16}$$

It is worth defining the terms in equation 16 in more detail,

- Q is the amount of charge transferred between the two plates of the capacitor. *Positive charge is transferred from the lower voltage plate, through the voltage source, to the higher voltage plate.*
- C is the capacitance, measured in farads, or coulombs per volt. It is the amount of charge transferred between the capacitor's plates when 1 volt is applied between them. As a coulomb of charge is a very large amount of charge, one farad is very large value of capacitance. You should therefore always expect your answers to be mF at the largest. Also note that **capacitance is a positive quantity**, therefore, always, *always*, check your answer is positive!
- V is the potential difference between the plates, with the higher voltage plate receiving the charge Q from the lower voltage plate.

1.5.2 Computing capacitance

A very common Tripos question is to derive an equation for the capacitance between two conducting objects, usually two parallel plates, two concentric cylinders, or two concentric spheres. For the simple capacitors that you will come across in the first year course, the capacitance C will be independent of the voltage, being purely determined by the structure of the capacitor - the size and shape of the plates, the distance between them, the dielectric constant of the separating material, etc..

Nearly all capacitance questions require you to follow the same procedure when answering,

1. Use Gauss's law to compute the electric field between the plates as function of charge and the capacitor structure. N.B. for most capacitors the electric field will be a function of the position between the plates

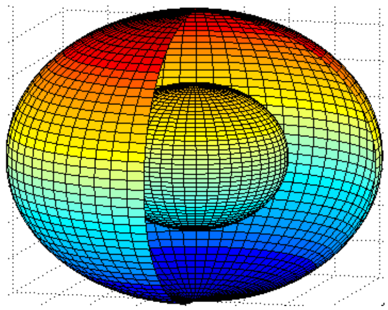

Figure 4: Two concentric spheres. The inner has radius r_1 and the outer r_2

2. Use the voltage equation to derive an expression for the potential difference between the plates as a function of charge and the capacitor structure (see section 1.4.1)
3. Substitute the derived equation for voltage into equation 16. The charge Q should cancel out leaving you with an expression just in terms of the capacitor's structure.

Common mistake: getting an expression for the electric field between the plates which is independent of position (i.e. a uniform electric field) when the true field varies with position.

Common mistake: not choosing the correct direction for the voltage difference, V. Consider figure 3, charge Q has been transferred from the bottom plate to the top plate, therefore, as explained above, $V = V_{\text{top}} - V_{\text{bottom}} = V_1 - V_2$. Note that this is the opposite way around to the voltage difference calculated in section 1.4.1, hence you will most likely need to negate your calculated voltage difference - *forgetting this is very common!*

Worked Example:

Calculate the capacitance between two concentric spheres of radius r_1 and r_2, with $r_1 < r_2$.

Step 1: Draw a diagram - We set up the problem as in figure 4, where the inner sphere has a radius r_1 and the outer sphere has radius r_2. We define Q to be the amount of charge transferred from the outer sphere to the inner sphere by the voltage source; hence the inner sphere has a charge of $+Q$ (and the outer sphere $-Q$ but this is irrelevant).

Step 2: Gauss's Law - We define an spherical Gaussian surface and apply Gauss's Law as in section 1.2. The electric field at a variable radius r is then, as previously derived in equation 5,

$$E = \frac{Q}{4\,\pi\,\epsilon\,r^2} \tag{17}$$

where $r_1 < r < r_2$.

Step 3: Potential difference - We need the potential difference between the inner and outer spheres. Given that positive charge has been transferred to the inner sphere from the outer, the potential difference that we require is $V = V_{inner} - V_{outer} = V(r_1) - V(r_2)$ (re-read section 1.5.1 if you're unsure why). Using the equation for electric potential difference (equation 14) as in section 1.4.1,

$$\begin{aligned} V(r_2) - V(r_1) &= -\int_{r_1}^{r_2} E(r)\,dr \\ &= -\int_{r_1}^{r_2} \frac{Q}{4\,\pi\,\epsilon\,r^2}\,dr \\ &= \left[\frac{Q}{4\,\pi\,\epsilon\,r}\right]_{r_1}^{r_2} \\ &= \frac{Q}{4\,\pi\,\epsilon}\left(\frac{1}{r_2} - \frac{1}{r_1}\right) \qquad (18) \\ \Rightarrow V &= \frac{Q}{4\,\pi\,\epsilon}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) \qquad (19) \end{aligned}$$

Step 4: Capacitance - Finally we apply equation 16,

$$C = \frac{Q}{V} = \frac{4\,\pi\,\epsilon}{\frac{1}{r_1} - \frac{1}{r_2}} \tag{20}$$

Note that $r_2 > r_1 \Rightarrow \frac{1}{r_1} > \frac{1}{r_2}$, and so the minus signs are correct as we have a positive value for capacitance. This came from remembering the minus sign in equation 8 and using the correct V.

1.5.3 The force between plates

Another common question is to find the force between the two plates of a capacitor. To do this we recall the equation for electrostatic force, $\boldsymbol{F} = \boldsymbol{E}q$. We need to take care when applying this equation however as we have two charged objects. A more rigorous statement of the equation would be,

The electrostatic force on object 2 resulting from the charge on object 1 is equal to the electric field strength produced by object 1 in isolation and measured at the location of object 2, multiplied by the charge on object 2.

The key point is that if we want the force on one plate due to the other plate then the E in the force equation is the electric field produced by one plate in isolation, not the combined electric field. Another way of thinking about this is that the q in the force equation is the charge on one plate and this charge cannot also be contributing to E - each charge can only appear in the equation once.

An example: find the force on the plates of a parallel plate capacitor in terms of the amount of charge transferred between the plates and their separation.

Let the amount of charge transferred from one plate to the other be Q; thus the charge on one plate is Q and the other $-Q$. We choose to calculate the force on the negatively charged plate, which implies we need to compute the electric field produced by the positively charged plate in isolation; we can do this by using Gauss's law. For a worked example of how to do this, see the *Charged Plates* document. The result is,

$$\boldsymbol{E} = \frac{Q}{2\,\epsilon\,A}\hat{\boldsymbol{r}} \tag{21}$$

where A is the area of a plate and $\hat{\boldsymbol{r}}$ is a unit vector pointing from the positive plate to the negative plate. We now plug this into the force equation using the charge on the negatively charged plate as q, i.e. $q = -Q$,

$$\begin{aligned} \boldsymbol{F} &= \boldsymbol{E}\,q \\ &= -\frac{Q^2}{2\,\epsilon\,A}\,\hat{\boldsymbol{r}} \end{aligned} \tag{22}$$

Force and electric field are both vector quantities, therefore the minus sign in the answer indicates that the force is in the opposite direction to the electric field. $\boldsymbol{E}$ points from the positively charged plate to the negatively charged plate and we have computed the force on the negatively charged plate. Therefore, the force points towards the positively charged plate, i.e. there is an attractive force between the plates.

1.6 The Method of Images

Consider the system in figure 5, where a charged object sits above a conducting plane, and imagine we are asked to find the electric field at some point below the sphere. The figure shows the field lines are no longer radial straight lines from the charged object. This means that a spherical Gaussian surface, centred on the charged object, will not be perpendicular to the $\boldsymbol{D}$-field. Thus applying Gauss's Law to solve this problem is significantly harder than to the charged object in isolation.

To solve this problem we can use the *method of images.* This states that the system on the left of figure 5 is identical in every way to the system on the right in figure 5. As the field lines must still be the same we still can't apply Gauss's Law to the whole system easily. However, we can use the law of superposition, which says that the electric field resulting from both objects is given by the sum of their individual contributions. This means that we can apply Gauss's Law, in the normal way, to each object in isolation. We can then combine their contributions at the point of interest.

In summary,

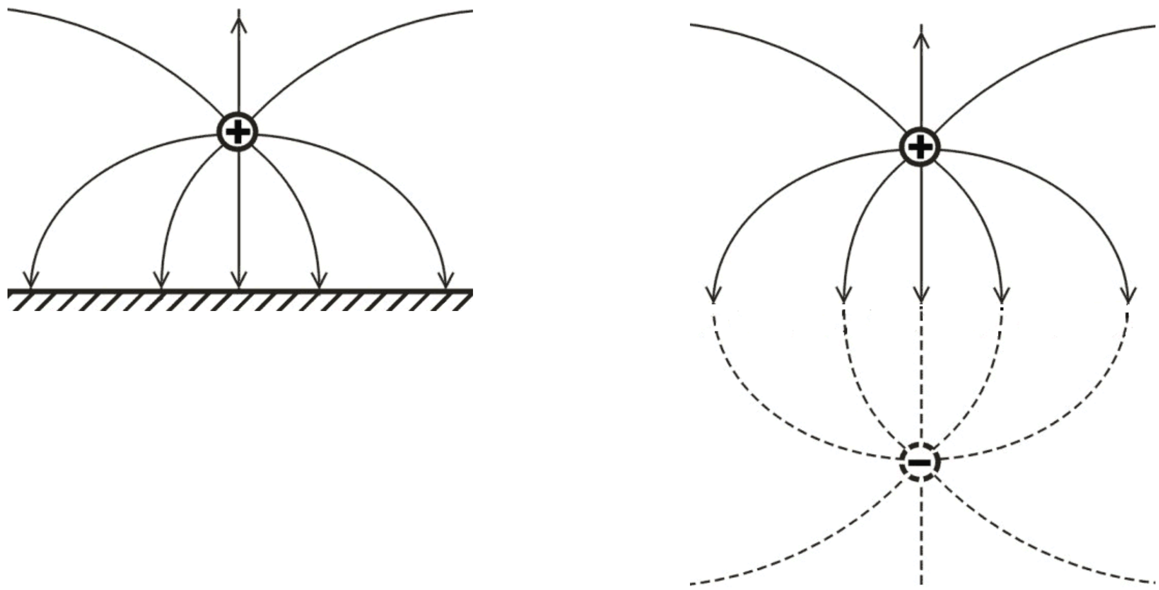

Figure 5: Left: A charged object above a conducting plane. Right: Applying the method of images to the problem. Both panes show identical setups.

1. Replace the conducting plane with a 'mirror image' charge
2. Use Gauss's law on each charge in isolation
3. Sum up the electric field contribution from each charge

2 Electromagnetism

2.1 Magnetic $\boldsymbol{H}$- and $\boldsymbol{B}$-fields

These are two *vector fields*, often both referred to as the magnetic field. In the Cambridge course $\boldsymbol{B}$ is most often known as the *magnetic flux density*. For linear materials, the two quantities are related by:

$$\boldsymbol{B} = \mu_0 \, \mu_r \, \boldsymbol{H}$$

where μ_0 is the *permeability of free space* and μ_r is the *relative permeability.* $\boldsymbol{B}$ does not depend on the material the magnetic field is in. Note, the product $\mu_0 \, \mu_r$ is often written as μ.

An important point to remember about both fields is that they are *linear.* This means they obey the rule of superposition. They are also both vector fields, which means that they have an associated direction.

2.2 Ampere's Law

Ampere's Law to magnetism is what Gauss's Law is to electrostatics. It is a key law, which you must understand.

Ampere's Law relates the integrated magnetic field around a closed loop to the electric current passing through the loop. Using Ampere's law, one can determine the magnetic field associated with a given current or current associated with a given magnetic field, providing there is no time changing electric field present. The law can be written in two forms, the 'integral form' and the 'differential form', and can be written in terms of either the $\boldsymbol{B}$- or $\boldsymbol{H}$-fields.

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} \;=\; \mu_0 \, I_{enc}$$
$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} \;=\; I_{enc} \tag{23}$$

The integrals can be done around any closed loop. However, they can be simplified if we choose our loop to be parallel with the magnetic field. In that case,

$$I_{enc} \;=\; \oint \boldsymbol{H} \cdot d\boldsymbol{l} \;=\; H \oint dl \;=\; HL \tag{24}$$

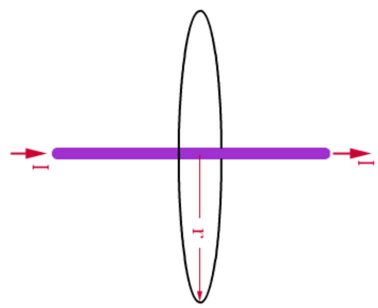

Figure 6: Application of Ampere's Law to a current carrying wire. The red loop illustrates our chosen Amperian loop. It is a circle centred on the wire as this ensures that the loop is parallel to the magnetic field lines.

Where L is the total length of the loop. Our choice of Amperian loop is therefore nearly always going to be a circle centred on the wire - remember that magnetic fields have circular field lines about a current carrying wire, and therefore, $\boldsymbol{H}$ and $d\boldsymbol{l}$ are always parallel for a circular loop centred on the wire. In which case,

$$I_{enc} \;=\; H \;\times\; 2\pi r \tag{25}$$

where r is the radius of the Amperian loop, a setup shown in figure 6. This enables us to calculate H at a particular distance r from a wire carrying current I by choosing our Amperian loop to have radius r. I.e. our Amperian loop passes through the point we are interested in.

Ampere's Law is sometimes used in situations where the current-carrying wire also forms a loop. Do not get confused, it is not this wire loop you integrate around but an imaginary loop around the wire, which passes through the point you are interested in. A typical case where this occurs is when analysing the magnetic field produced by a coil of wire. In above equations I have written I_enc to indicate that it is the total enclosed current we need. If our Amperian loop surrounds a coil of N turns carrying current I then the *total* enclosed current is NI, not just I.

An example:

A wire is bent into a square of side a and a current I is passed through the wire. Calculate the $\boldsymbol{B}$*-field at the center of the square.*

$\boldsymbol{B}$-fields are linear and so we can use superposition to calculate the total $\boldsymbol{B}$-field from first just considering each side on its own. We define our 'Amperian loop' as a circle about one side of the square of radius $a/2$ (thus it passes through the center of the wire square). The $\boldsymbol{B}$-field and the loop element $d\boldsymbol{l}$ are always parallel and so the dot product is $B\ dl$. We also know that, at a constant distance from the wire, the $\boldsymbol{B}$-field has a constant magnitude. Therefore:

$$\oint \boldsymbol{B} \cdot d\boldsymbol{l} \;=\; B \oint \; dl \;=\; B\, 2\pi \frac{a}{2} \;=\; \mu_0 \, I_{enc}$$
$$\Rightarrow \; B \;=\; \frac{\mu_0 \, I}{\pi \, a}$$

As there are four identical sides the final answer is,

$$B = \frac{4\,\mu_0\,I}{\pi\,a}$$

2.3 Faraday's Law

Faraday's law states that: The induced electromotive force (emf) in any closed circuit is equal to the rate of change of the magnetic flux through the circuit. The law strictly holds only when the closed circuit is an infinitely-thin wire. The equation is,

$$\text{emf} = -\frac{d\phi}{dt} \tag{26}$$

The 'emf' is a misnomer, it is not a force at all. There are numerous definitions of emf, many of which are not completely consistent with each other. A common definition is that the emf is the external work per unit charge, which must be done to separate a positive and negative charge in the circuit. That is, emf is the work done, per unit charge, to create a potential difference (voltage). Both emf and voltage have the same units of volts, or joules per coulomb, and the two terms are often used synonymously. The minus sign is a direct result of *Lenz's Law*, which states that the direction of induced current is such that the magnetic field it produces opposes the original inducing magnetic field.

If the magnetic field is (approximately) uniform over the wire loop then, remembering that $\boldsymbol{B}$ is the magnetic flux density, we can write the formula as,

$$\text{emf} = -\frac{d(\boldsymbol{B}\cdot\boldsymbol{A})}{dt}$$

where the direction of $\boldsymbol{A}$ is normal to the plane of the wire loop. Because the fields are linear the total EMF induced in multiple coils of wire is additive:

$$\text{emf} = -N\,\frac{d(\boldsymbol{B}\cdot\boldsymbol{A})}{dt}$$

Note: if $\boldsymbol{B}$ cannot be considered uniform over the cross-section of the circuit we must integrate $\boldsymbol{B}$ over the cross-section to find the total flux passing through the circuit.

An example:

A circular loop of wire of radius a is placed in a $\boldsymbol{B}$*-field, perpendicular, and 'face-on', to the field lines. The loop is then rotated at a constant angular velocity* ω*. Find the induced emf.*

First we replace the dot product with the product of the magnitudes and cosine of the angle between them. The magnetic field and the magnitude of the area of the loop are not changing so we can take them outside of the derivative.

$$\text{emf} = -B\,A\,\frac{d\cos(\theta)}{dt}$$

The angle between the normal to the plane containing the wire loop (the direction of $\boldsymbol{A}$) and the direction of the $\boldsymbol{B}$-field can be written in terms of the angular velocity and time. In our case the loop started 'face-on' to the magnetic field so we don't need to add an initial angle term.

$$\text{emf} = -B\,A\,\frac{d\cos(\omega t)}{dt} = B\,A\,\omega\,\sin(\omega t)$$

2.4 The Lorentz Force

The Lorentz force is the force on a point charge due to both electric and magnetic fields. It is given by,

$$\boldsymbol{F} = q\,[\boldsymbol{E} + (\boldsymbol{v}\times\boldsymbol{B})] \tag{27}$$

where $\boldsymbol{v}$ is the velocity of the charge. We can extend this to find a useful equation for the force on a wire of length L, carrying a current I, in an magnetic field. First we write:,

$$\boldsymbol{F} = Q\,\boldsymbol{v}\times\boldsymbol{B}$$

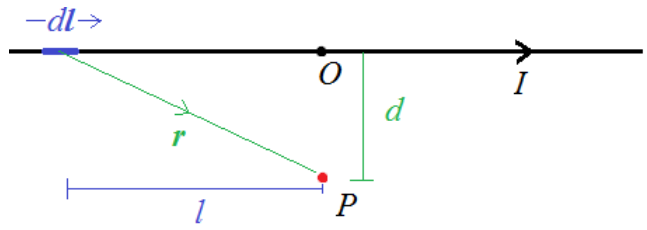

Figure 7: Biot-Savart Law applied to infinite wire

where Q is now the total amount of charge in the part of the wire inside the $\boldsymbol{B}$-field. Recall that $Q = I\,t$, that is a current I carries a charge Q passed a point in time t. In our case, this is equivalent to the time it takes an electron just entering the magnetic field to transverse the field and exit. Given that v is the speed of an electron we can combine these to give,

$$\boldsymbol{F} = L\,\boldsymbol{I} \times \boldsymbol{B}$$

which has a magnitude of,

$$|\boldsymbol{F}| = B\,I\,L\,\sin(\theta)$$

where L is the length of the wire and θ is the angle between the direction of the wire and the direction of the $\boldsymbol{B}$-field. Note that it is the component of the current flowing in the direction perpendicular to the $\boldsymbol{B}$-field that matters - charge crossing field lines.

2.5 The Biot-Savart Law

The Biot-Savart law is used to compute the magnetic field generated by a steady current, i.e. a continual flow of charges, for example through a wire, which is constant in time and in which charge is neither building up nor depleting at any point.

$$\boldsymbol{B} = \int \frac{\mu_0\,I}{4\pi r^2}\,d\boldsymbol{l} \times \hat{\boldsymbol{r}} \tag{28}$$

where,

- $\boldsymbol{B}$ is the magnetic field density (a vector)
- $d\boldsymbol{l}$ is the differential element of the wire in the direction of conventional current (a vector)
- r is the distance from the wire to the point at which the magnetic field is being calculated
- $\hat{\boldsymbol{r}}$ is the **unit** vector from the wire element to the point at which the magnetic field is being calculated

An example:

Calculate the magnetic field at point P, which is at a distance d from an infinitely long wire, carrying current I.

Define the point above P on the wire as the origin, and the x-axis as the co-ordinate l. $\hat{\boldsymbol{r}}$ is a unit vector from the element $d\boldsymbol{l}$, at a distance l from the origin along the wire, to the point P. The magnitude of the cross product between two vectors is the product of their magnitudes multiplied by the sine of the angle between them:

$$|d\boldsymbol{l} \times \hat{\boldsymbol{r}}| = |d\boldsymbol{l}||\hat{\boldsymbol{r}}|\sin\theta = dl\,\frac{d}{r}$$

We also know that:

$$r = (l^2 + d^2)^{\frac{1}{2}}$$

Using the right-hand grip rule we can see that the B-field at point P will be into the page. We define a unit vector in this direction as $\hat{\boldsymbol{u}}$. Therefore:

$$
\begin{aligned}
\boldsymbol{B} &= \frac{\mu_0 \; I \; d}{4\pi} \int_{-\infty}^{\infty} (l^2 \; + \; d^2)^{-\frac{3}{2}} \; dl \; \hat{\boldsymbol{u}} \\
&= \frac{\mu_0 \; I \; d}{4\pi} \left[l \; d^{-2} (l^2 \; + \; d^2)^{-\frac{1}{2}} \right]_{-\infty}^{\infty} \; \hat{\boldsymbol{u}} \\
&= \frac{\mu_0 \; I}{2\pi d} \; \hat{\boldsymbol{u}}
\end{aligned}
$$

Compare this with Ampere's Law.