Are You a Bayesian or a Frequentist?

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Statistical Inference

- Bayesian perspective
  - conditional perspective—inferences should be made conditional on the current data
  - natural in the setting of a long-term project with a domain expert
  - the optimist—let's make the best possible use of our sophisticated inferential tool

- Frequentist perspective
  - unconditional perspective—inferential methods should give good answers in repeated use
  - natural in the setting of writing software that will be used by many people with many data sets
  - the pessimist—let's protect ourselves against bad decisions given that our inferential procedure is inevitably based on a simplification of reality

Machine Learning (As Explained to a Statistician)

- A loose confederation of themes in statistical inference (and decision-making)
- A focus on prediction and exploratory data analysis
  - not much worry about "coverage"
- A focus on computational methodology and empirical evaluation, with a dollop of empirical process theory
  - lots of nonparametrics, but not much asymptotics
- Sometimes Bayesian and sometimes frequentist
  - not much interplay

Decision-Theoretic Perspective

- Define a family of probability models for the data X, indexed by a "parameter" θ
- Define a "procedure" $\delta(X)$ that operates on the data to produce a decision
- Define a loss function:
$$l(\delta(X), \theta)$$
- The goal is to use the loss function to compare procedures, but both of its arguments are unknown

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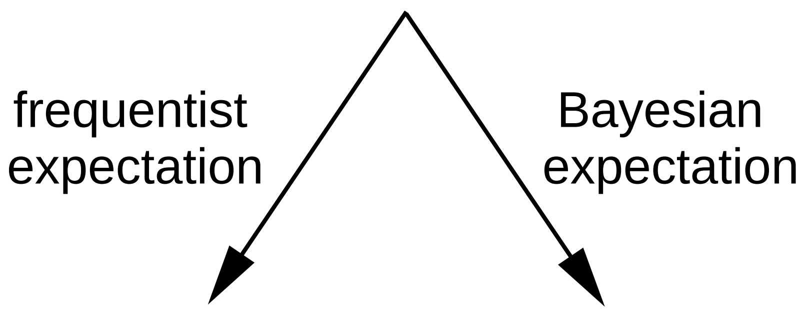

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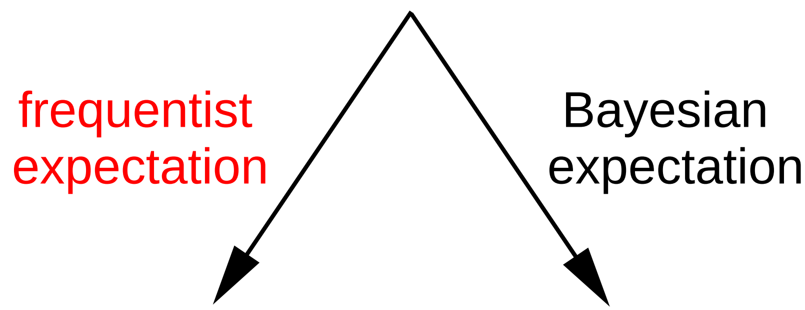

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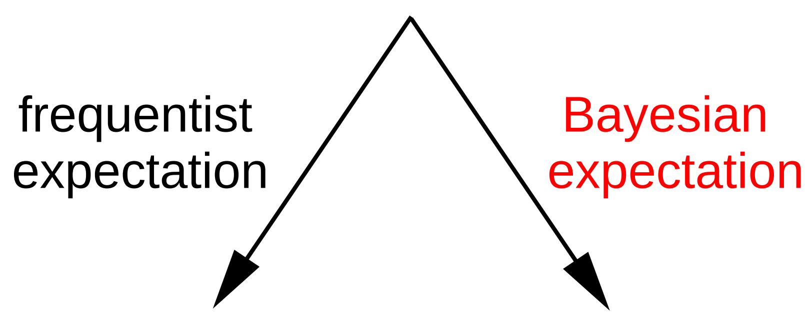

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Coherence and Calibration

- Coherence and calibration are two important goals for statistical inference
- Bayesian work has tended to focus on coherence while frequentist work hasn't been too worried about coherence
  - the problem with pure coherence is that one can be coherent and completely wrong
- Frequentist work has tended to focus on calibration while Bayesian work hasn't been too worried about calibration
  - the problem with pure calibration is that one can be calibrated and completely useless
- Many statisticians find that they make use of both the Bayesian perspective and the frequentist perspective, because a blend is often a natural way to achieve both coherence and calibration

The Bayesian World

- The Bayesian world is further subdivided into subjective Bayes and objective Bayes

- Subjective Bayes: work hard with the domain expert to come up with the model, the prior and the loss

- Subjective Bayesian research involves (inter alia) developing new kinds of models, new kinds of computational methods for integration, new kinds of subjective assessment techniques

- Not much focus on analysis, because the spirit is that "Bayes is optimal" (given a good model, a good prior and a good loss)

Subjective Bayes

- A fairly unassailable framework in principle, but there are serious problems in practice:
  - for complex models, there can be many, many unknown parameters whose distributions must be assessed
  - independence assumptions often must be imposed to make it possible for humans to develop assessments
  - independence assumptions often must be imposed to obtain a computationally tractable model
  - it is particularly difficult to assess tail behavior, and tail behavior can matter (cf. marginal likelihoods and Bayes factors)
  - Bayesian nonparametrics can be awkward for subjective Bayes

- Also, there are lots of reasonable methods out there that don't look Bayesian; why should we not consider them?

Objective Bayes

- When the subjective Bayesian runs aground in complexity, the objective Bayesian attempts to step in

- The goal is to find principles for setting priors so as to have minimal impact on posterior inference

- E.g., reference priors maximize the divergence between the prior and the posterior
  - which often yields "improper priors"

- Objective Bayesians often make use of frequentist ideas in developing principles for choosing priors

- An appealing framework (and a great area to work in), but can be challenging to work with in complex (multivariate, hierarchical) models

Frequentist Perspective

- From the frequentist perspective, procedures can come from anywhere; they don't have to be derived from a probability model
  - e.g., nonparametric testing
  - e.g., the support vector machine, boosting
  - e.g., methods based on first-order logic
- This opens the door to some possibly silly methods, so it's important to develop principles and techniques of analysis that allow one to rule out methods, and to rank the reasonable methods
- Frequentist statistics tends to focus more on analysis than on methods
- (One general method—the bootstrap)

Frequentist Activities

- There is a hierarchy of analytic activities:
  - consistency
  - rates
  - sampling distributions

- Classical frequentist statistics focused on parametric statistics, then there was a wave of activity in nonparametric testing, and more recently there has been a wave of activity in other kinds of nonparametrics
  - e.g., function estimation
  - e.g., large p, small n problems

- One of the most powerful general tools is empirical process theory, where consistency, rates and sampling distributions are obtained uniformly on various general spaces (this is the general field that encompasses statistical learning theory)

Outline

- Surrogate loss functions, f-divergences and experimental design
- Composite loss functions and multivariate regression
- Sufficient dimension reduction
- Sparse principal component analysis

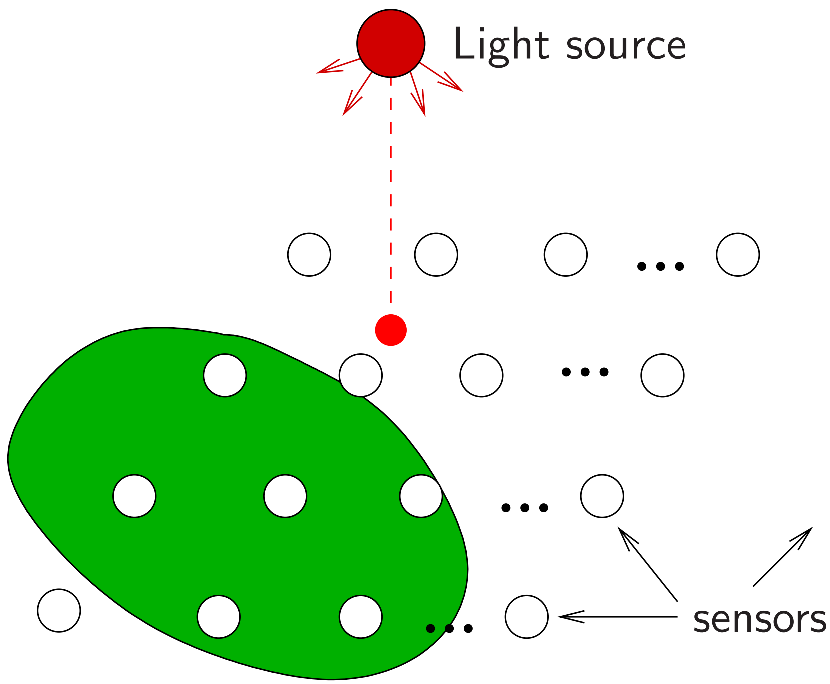
Surrogate Loss Functions, f-Divergences and Experimental Design

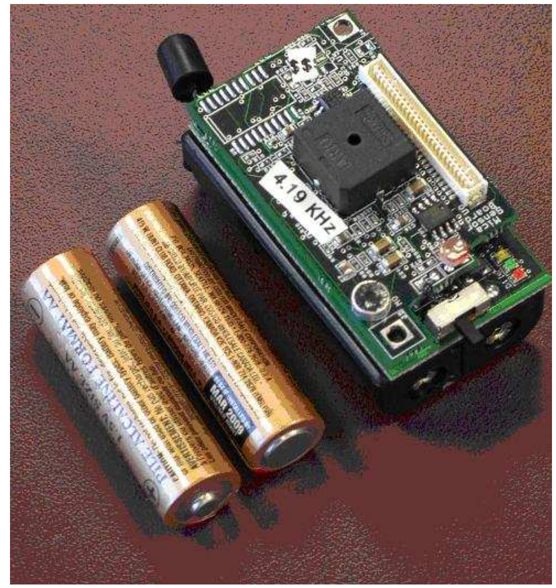
Nguyen, X., Wainwright, M. J., and Jordan, M. I. (2008). On loss functions and f-divergences. *Annals of Statistics, 37*, 876–904.

Bartlett, P., Jordan, M. I., and McAuliffe, J. (2006). Convexity, classification and risk bounds. *Journal of the American Statistical Association, 101*, 138–156.

Nguyen, X., Jordan, M. I., and Sinopoli, B. (2005). *ACM Transactions on Sensor Networks, 1*, 134–152.

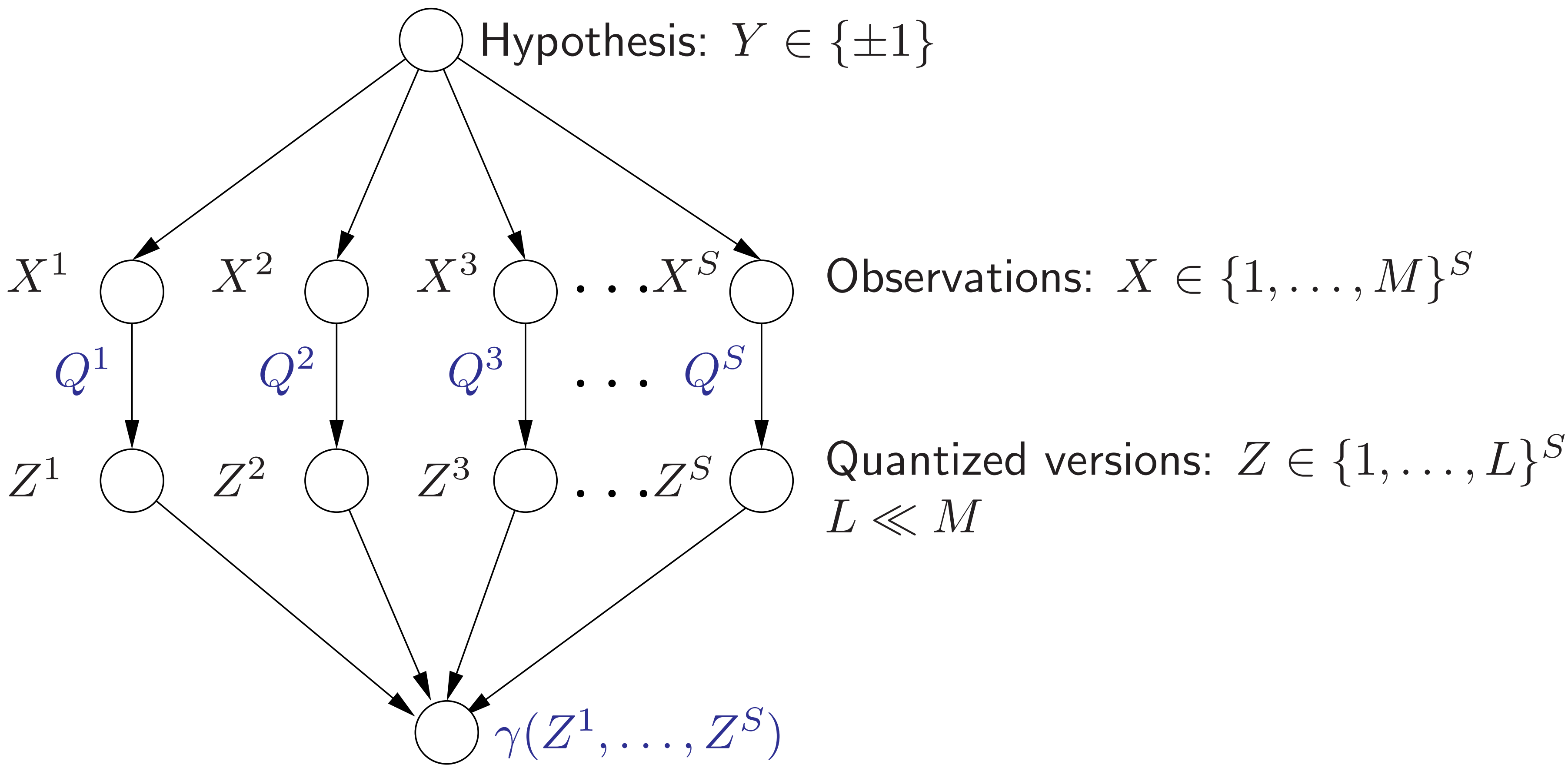
Motivating Example: Decentralized Detection





- Wireless network of motes equipped with sensors (e.g., light, heat, sound)
- Limited battery: can only transmit quantized observations
- Is the light source above the green region?

Decentralized Detection



Decentralized Detection (cont.)

- General set-up:
  - data are (X, Y) pairs, assumed sampled i.i.d. for simplicity, where $Y \in \{0, 1\}$
  - given X, let $Z = Q(X)$ denote the covariate vector, where $Q \in \mathcal{Q}$, where $\mathcal{Q}$ is some set of random mappings (can be viewed as an experimental design)
  - consider a family $\{\gamma(\cdot)\}$, where γ is a discriminant function lying in some (nonparametric) family Γ

- Problem: Find the decision $(Q; \gamma)$ that minimizes the probability of error $P(Y \neq \gamma(Z))$

- Applications include:
  - decentralized compression and detection
  - feature extraction, dimensionality reduction
  - problem of sensor placement

Perspectives

- *Signal processing literature*
  - everything is assumed known except for Q—the problem of "decentralized detection" is to find Q
  - this is done via the maximization of an "f-divergence" (e.g., Hellinger distance, Chernoff distance)
  - basically a heuristic literature from a statistical perspective (plug-in estimation)

- *Statistical machine learning literature*
  - Q is assumed known and the problem is to find γ
  - this is done via the minimization of an "surrogate loss function" (e.g., boosting, logistic regression, support vector machine)
  - decision-theoretic flavor; consistency results

f-divergences (Ali-Silvey Distances)

The f-divergence between measures μ and π is given by

$$I_f(\mu,\pi) := \sum_z \pi(z) f\left(\frac{\mu(z)}{\pi(z)}\right).$$

where $f : [0,+\infty) \to \mathbb{R} \cup \{+\infty\}$ is a continuous convex function

- Kullback-Leibler divergence: $f(u) = u \log u$.

$$I_f(\mu,\pi) = \sum_z \mu(z) \log \frac{\mu(z)}{\pi(z)}.$$

- variational distance: $f(u) = |u-1|$.

$$I_f(\mu,\pi) := \sum_z |\mu(z) - \pi(z)|.$$

- Hellinger distance: $f(u) = \frac{1}{2}(\sqrt{u}-1)^2$.

$$I_f(\mu,\pi) := \sum_{z\in\mathcal{Z}} (\sqrt{\mu(z)} - \sqrt{\pi(z)})^2.$$

Why the f-divergence?

- A classical theorem due to Blackwell (1951): *If a procedure A has a smaller f-divergence than a procedure B (for some fixed f), then there exist some set of prior probabilities such that procedure A has a smaller probability of error than procedure B*

- Given that it is intractable to minimize probability of error, this result has motivated (many) authors in signal processing to use f-divergences as surrogates for probability of error

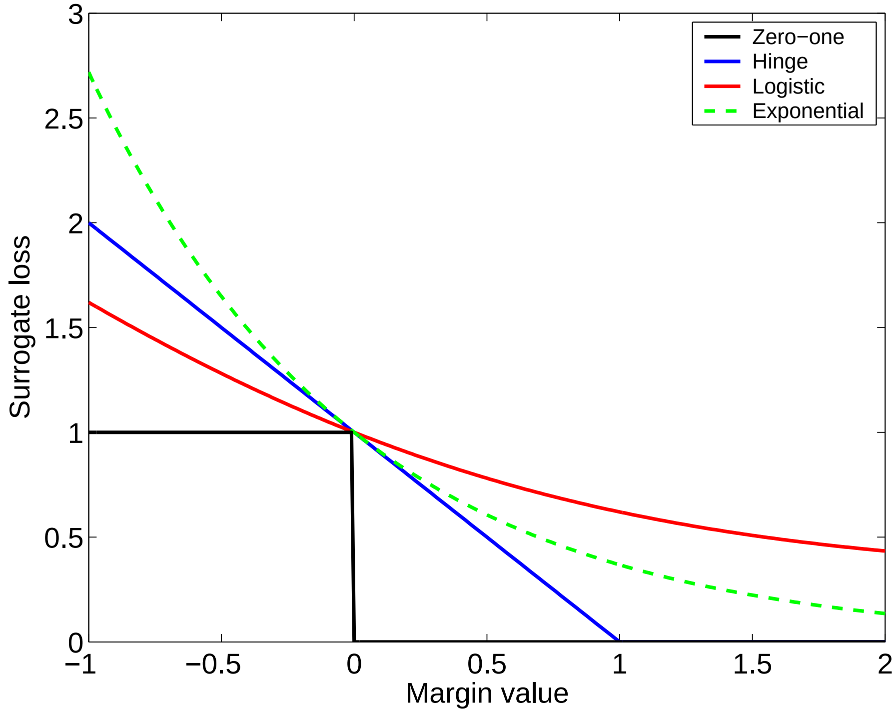
- I.e., choose a quantizer Q by maximizing an f-divergence between $P(Z|Y = 1)$ and $P(Z|Y = -1)$
  - Hellinger distance (Kailath 1967; Longo et al, 1990)
  - Chernoff distance (Chamberland & Veeravalli, 2003)

- Supporting arguments from asymptotics
  - Kullback-Leibler divergence in the Neyman-Pearson setting
  - Chernoff distance in the Bayesian setting

Statistical Machine Learning Perspective

- *Decision-theoretic*: based on a loss function $\phi(Y, \gamma(Z))$
- E.g., 0-1 loss:
$$\phi(Y, \gamma(Z)) = \begin{cases} 1 & \text{if } Y \neq \gamma(Z) \\ 0 & \text{otherwise} \end{cases}$$
which can be written in the binary case as $\phi(Y, \gamma(Z)) = \mathbb{I}(Y\gamma(Z) < 0)$
- The main focus is on estimating γ; the problem of estimating Q by minimizing the loss function is only occasionally addressed
- It is intractable to minimize 0-1 loss, so consider minimizing a surrogate loss functions that is a convex upper bound on the 0-1 loss

Margin-Based Surrogate Loss Functions



- Define a convex surrogate in terms of the margin $u = y\gamma(z)$
  - hinge loss: $\phi(u) = \max(0, 1-u)$ — support vector machine
  - exponential loss: $\phi(u) = \exp(-u)$ — boosting
  - logistic loss: $\phi(u) = \log[1 + \exp(-u)]$ — logistic regression

Estimation Based on a Convex Surrogate Loss

- Estimation procedures used in the classification literature are generally M-estimators ("empirical risk minimization")

- Given i.i.d. training data $(x_1, y_1), \ldots, (x_n, y_n)$

- Find a classifier γ that minimizes the empirical expectation of the surrogate loss:

$$\hat{\mathbb{E}}\phi(Y\gamma(X)) := \frac{1}{n}\sum_{i=1}^{n}\phi(y_i\gamma(x_i))$$

  where the convexity of ϕ makes this feasible in practice and in theory

Some Theory for Surrogate Loss Functions

(Bartlett, Jordan, & McAuliffe, JASA 2006)

- ϕ must be classification-calibrated, i.e., for any $a, b \geq 0$ and $a \neq b$,

$$\inf_{\alpha:\alpha(a-b)<0} \phi(\alpha)a + \phi(-\alpha)b > \inf_{\alpha \in \mathbb{R}} \phi(\alpha)a + \phi(-\alpha)b$$

  (essentially a form of Fisher consistency that is appropriate for classification)

- This is necessary and sufficient for Bayes consistency; we take it as the definition of a "surrogate loss function" for classification

- In the convex case, ϕ is classification-calibrated *iff* differentiable at 0 and $\phi'(0) < 0$

Outline

- A precise link between surrogate convex losses and f-divergences
  - we establish a constructive and many-to-one correspondence
- A notion of universal equivalence among convex surrogate loss functions
- An application: Proof of consistency for the choice of a (Q, γ) pair using any convex surrogate for the 0-1 loss

Setup

- We want to find (Q, γ) to minimize the ϕ-*risk*

$$R_\phi(\gamma, Q) = \mathbb{E}\phi(Y\gamma(Z))$$

- Define:

$$\begin{aligned}
\mu(z) &= P(Y=1, z) = p \int_x Q(z|x) dP(x|Y=1) \\
\pi(z) &= P(Y=-1, z) = q \int_x Q(z|x) dP(x|Y=-1).
\end{aligned}$$

- ϕ-risk can be represented as:

$$R_\phi(\gamma, Q) = \sum_z \phi(\gamma(z))\mu(z) + \phi(-\gamma(z))\pi(z)$$

Profiling

- Optimize out over γ (for each z) and define:

$$R_\phi(Q) := \inf_{\gamma\in\Gamma} R_\phi(\gamma, Q)$$

- For example, for 0-1 loss, we easily obtain $\gamma(z) = \mathrm{sign}(\mu(z) - \pi(z))$. Thus:

$$\begin{aligned} R_{\text{0-1}}(Q) &= \sum_{z\in\mathcal{Z}} \min\{\mu(z), \pi(z)\} \\ &= \frac{1}{2} - \frac{1}{2}\sum_{z\in\mathcal{Z}} |\mu(z) - \pi(z)| \\ &= \frac{1}{2}(1 - V(\mu, \pi)) \end{aligned}$$

  where $V(\mu, \pi)$ is the variational distance.

- I.e., optimizing out a ϕ-risk yields an f-divergence. Does this hold more generally?

Some Examples

- **hinge loss**:

$$R_{hinge}(Q) = 1 - V(\mu, \pi)$$

(variational distance)

- **exponential loss**:

$$R_{exp}(Q) = 1 - \sum_{z \in \mathcal{Z}} (\sqrt{\mu(z)} - \sqrt{\pi(z)})^2$$

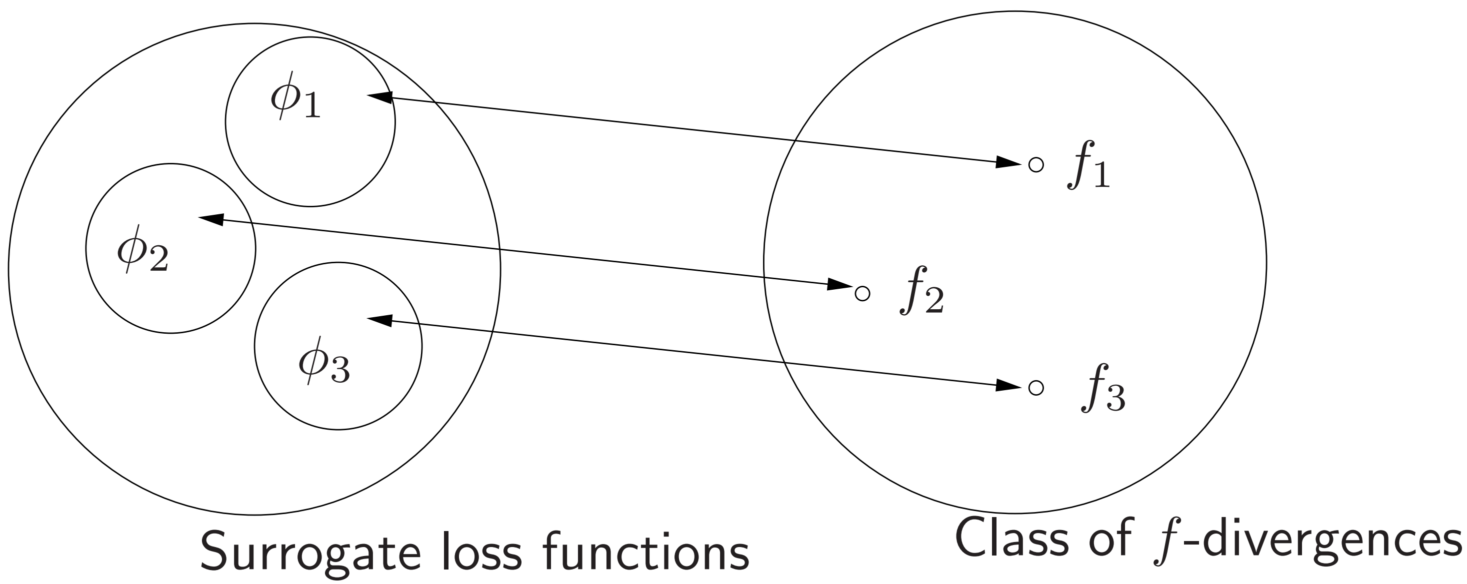
(Hellinger distance)

- **logistic loss**:

$$R_{log}(Q) = \log 2 - D(\mu \| \frac{\mu + \pi}{2}) - D(\pi \| \frac{\mu + \pi}{2})$$

(capacitory discrimination)

Link between ϕ-losses and f-divergences



Conjugate Duality

- Recall the notion of *conjugate duality* (Rockafellar): For a lower-semicontinuous convex function $f : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$, the conjugate dual $f^* : \mathbb{R} \longrightarrow \mathbb{R} \cup \{\infty\}$ is defined as

$$f^*(u) = \sup_{v \in \mathbb{R}} \{uv - f(v)\},$$

  which is necessarily a convex function.

- Define

$$\Psi(\beta) = f^*(-\beta)$$

Link between ϕ-losses and f-divergences

Theorem 1. *(a) For any margin-based surrogate loss function ϕ, there is an f-divergence such that $R_\phi(Q) = -I_f(\mu, \pi)$ for some lower-semicontinuous convex function f.*

In addition, if ϕ is continuous and satisfies a (weak) regularity condition, then the following properties hold:

(i) Ψ *is a decreasing and convex function.*

(ii) $\Psi(\Psi(\beta)) = \beta$ *for all* $\beta \in (\beta_1, \beta_2)$.

(iii) There exists a point u^* *such that* $\Psi(u^*) = u^*$.

(b) Conversely, if f is a lower-semicontinuous convex function satisfying conditions (i–iii), there exists a decreasing convex surrogate loss ϕ that induces the corresponding f-divergence

The Easy Direction: $\phi \to f$

- Recall

$$R_\phi(\gamma, Q) = \sum_{z \in \mathcal{Z}} \phi(\gamma(z))\mu(z) + \phi(-\gamma(z))\pi(z)$$

- Optimizing out $\gamma(z)$ for each z:

$$R_\phi(Q) = \sum_{z \in \mathcal{Z}} \inf_\alpha \phi(\alpha)\mu(z) + \phi(-\alpha)\pi(z) = \sum_z \pi(z) \inf_\alpha \left(\phi(-\alpha) + \phi(\alpha)\frac{\mu(z)}{\pi(z)} \right)$$

- For each z let $u = \frac{\mu(z)}{\pi(z)}$, define:

$$f(u) := -\inf_\alpha (\phi(-\alpha) + \phi(\alpha)u)$$

  - f is a convex function
  - we have

$$R_\phi(Q) = -I_f(\mu, \pi)$$

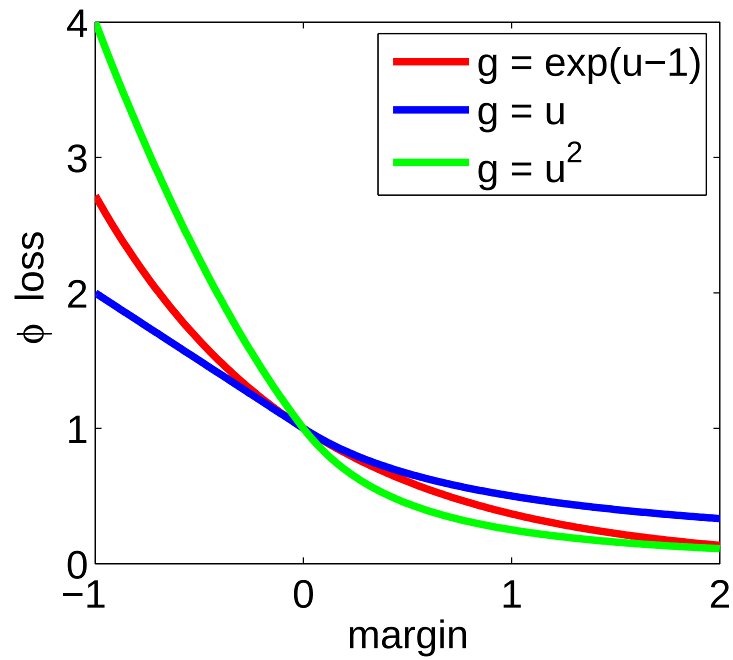
The $f \to \phi$ Direction Has a Constructive Consequence

- Any continuous loss function ϕ that induces an f-divergence must be of the form

$$\phi(\alpha) = \begin{cases} u^* & \text{if } \alpha = 0 \\ \Psi(g(\alpha + u^*)) & \text{if } \alpha > 0 \\ g(-\alpha + u^*) & \text{if } \alpha < 0, \end{cases}$$

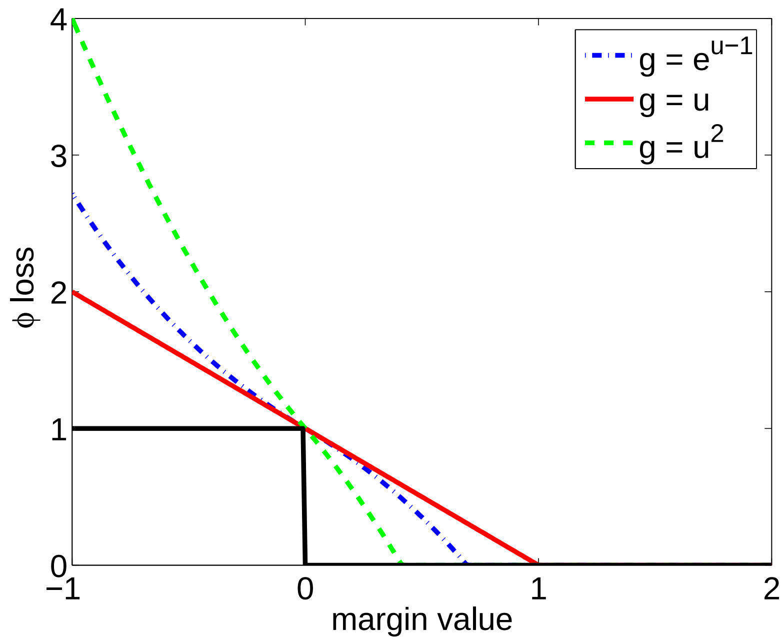
where $g : [u^*, +\infty) \to \overline{\mathbb{R}}$ is some increasing continuous and convex function such that $g(u^*) = u^*$, and g is right-differentiable at u^* with $g'(u^*) > 0$.

Example – Hellinger distance



- Hellinger distance corresponds to an f-divergence with $f(u) = -2\sqrt{u}$
- Recover immediate function $\Psi(\beta) = f^*(-\beta) = \begin{cases} 1/\beta & \text{when } \beta > 0 \\ +\infty & \text{otherwise.} \end{cases}$
- Choosing $g(u) = e^{u-1}$ yields $\phi(\alpha) = \exp(-\alpha)$ $\Rightarrow$ exponential loss

Example – Variational distance

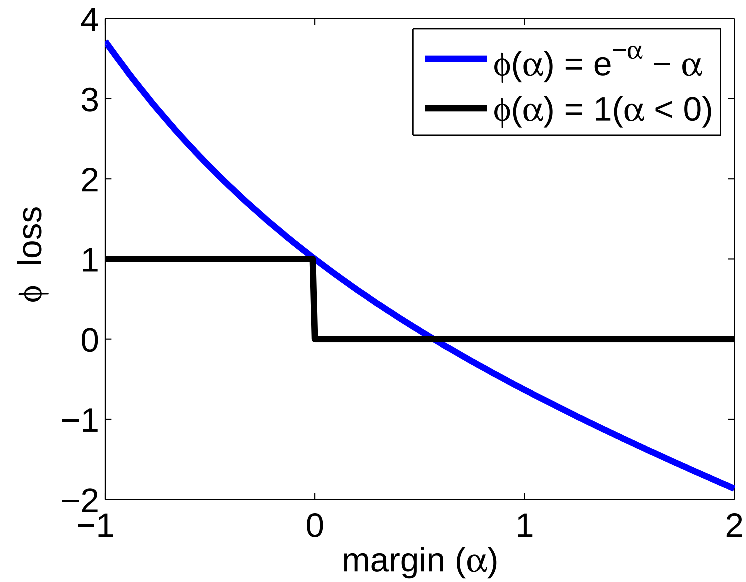


- Variational distance corresp. to an f-divergence with $f(u) = -2\min\{u, 1\}$

- Recover immediate function $\Psi(\beta) = f^*(-\beta) = \begin{cases} (2-\beta)_+ & \text{when } \beta > 0 \\ +\infty & \text{otherwise.} \end{cases}$

- Choosing $g(u) = u$ yields $\phi(\alpha) = (1-\alpha)_+$ $\Rightarrow$ hinge loss

Example – Kullback-Leibler divergence



- There is no corresponding ϕ loss for either $D(\mu\|\pi)$ or $D(\pi\|\mu)$
- But the *symmetrized* KL divergence $D(\mu\|\pi) + D(\pi\|\mu)$ is realized by

$$\phi(\alpha) = e^{-\alpha} - \alpha$$

Bayes Consistency for Choice of (Q, λ)

- Recall that from the 0-1 loss, we obtain the variational distance as the corresponding f-divergence, where $f(u) = \min\{u, 1\}$.

- Consider a broader class of f-divergences defined by:

$$f(u) = -c \min\{u, 1\} + au + b$$

- And consider the set of (continuous, convex and classification-calibrated) ϕ-losses that can be obtained (via Theorem 1) from these f-divergences

- We will provide conditions under which such ϕ-losses yield Bayes consistency for procedures that jointly choose (Q, λ)

- (And later we will show that *only* such ϕ-losses yield Bayes consistency)

Setup

- Consider sequences of increasing compact function classes $\mathcal{C}_1 \subseteq \ldots \subseteq \Gamma$ and $\mathcal{D}_1 \subseteq \ldots \subseteq \mathcal{Q}$

- Assume there exists an oracle that outputs an optimal solution to:

$$\min_{(\gamma,Q)\in(\mathcal{C}_n,\mathcal{D}_n)} \hat{R}_\phi(\gamma, Q) = \min_{(\gamma,Q)\in(\mathcal{C}_n,\mathcal{D}_n)} \frac{1}{n}\sum_{i=1}^{n}\sum_{z\in\mathcal{Z}} \phi(Y_i\gamma(z))Q(z|X_i)$$

  and let (γ_n^*, Q_n^*) denote one such solution.

- Let R^*_{Bayes} denote the minimum Bayes risk:

$$R^*_{Bayes} := \inf_{(\gamma,Q)\in(\Gamma,\mathcal{Q})} R_{Bayes}(\gamma, Q).$$

- Excess Bayes risk: $R_{Bayes}(\gamma_n^*, Q_n^*) - R^*_{Bayes}$

Setup

- *Approximation error*:

$$\mathcal{E}_0(\mathcal{C}_n, \mathcal{D}_n) = \inf_{(\gamma, Q) \in (\mathcal{C}_n, \mathcal{D}_n)} \{R_\phi(\gamma, Q)\} - R_\phi^*$$

where $R_\phi^* := \inf_{(\gamma, Q) \in (\Gamma, \mathcal{Q})} R_\phi(\gamma, Q)$

- *Estimation error*:

$$\mathcal{E}_1(\mathcal{C}_n, \mathcal{D}_n) = \mathbb{E} \sup_{(\gamma, Q) \in (\mathcal{C}_n, \mathcal{D}_n)} \left| \hat{R}_\phi(\gamma, Q) - R_\phi(\gamma, Q) \right|$$

where the expectation is taken with respect to the measure $\mathbb{P}^n(X, Y)$

Bayes Consistency for Choice of (Q, λ)

Theorem 2.

Under the stated conditions:

$$R_{Bayes}(\gamma_n^*, Q_n^*) - R_{Bayes}^* \quad \leq \quad \frac{2}{c}\left\{2\mathcal{E}_1(\mathcal{C}_n, \mathcal{D}_n) + \mathcal{E}_0(\mathcal{C}_n, \mathcal{D}_n) + 2M_n\sqrt{2\frac{\ln(2/\delta)}{n}}\right\}$$

- Thus, under the usual kinds of conditions that drive approximation and estimation error to zero, and under the additional condition on ϕ:

$$M_n \;:=\; \max_{y\in\{-1,+1\}} \; \sup_{(\gamma,Q)\in(\mathcal{C}_n,\mathcal{D}_n)} \; \sup_{z\in\mathcal{Z}} |\phi(y\gamma(z))| \;<\; +\infty,$$

  we obtain Bayes consistency (for the class of ϕ obtained from $f(u) = -c\min\{u, 1\} + au + b$)

Universal Equivalence of Loss Functions

- Consider two loss functions ϕ_1 and ϕ_2, corresponding to f-divergences induced by f_1 and f_2
- ϕ_1 and ϕ_2 are **universally** equivalent, denoted by

$$\phi_1 \overset{u}{\approx} \phi_2$$

if for **any** $P(X,Y)$ and quantization rules Q_A, Q_B, there holds:

$$R_{\phi_1}(Q_A) \leq R_{\phi_1}(Q_B) \Leftrightarrow R_{\phi_2}(Q_A) \leq R_{\phi_2}(Q_B).$$

An Equivalence Theorem

Theorem 3.
$$\phi_1 \overset{u}{\approx} \phi_2$$
if and only if
$$f_1(u) = cf_2(u) + au + b$$
for constants $a, b \in \mathbb{R}$ and $c > 0$.

- $\Leftarrow$ is easy; $\Rightarrow$ is not

- In particular, surrogate losses universally equivalent to 0-1 loss are those whose induced f divergence has the form:

$$f(u) = -c\min\{u, 1\} + au + b$$

- Thus we see that *only* such losses yield Bayes consistency for procedures that jointly choose (Q, λ)

Estimation of Divergences

- Given i.i.d. $\{x_1, \ldots, x_n\} \sim \mathbb{Q}$, $\{y_1, \ldots, y_n\} \sim \mathbb{P}$
  - $\mathbb{P}, \mathbb{Q}$ are unknown multivariate distributions with densities p_0, q_0 wrt Lesbegue measure μ on $\mathbb{R}^d$

- Consider the problem of estimating a divergence; e.g., KL divergence:
  - Kullback-Leibler (KL) divergence functional

$$D_K(\mathbb{P}, \mathbb{Q}) = \int p_0 \log \frac{p_0}{q_0} \, d\mu$$

Existing Work

- Relations to entropy estimation
  - large body of work on functional of one density (Bickel & Ritov, 1988; Donoho & Liu 1991; Birgé & Massart, 1993; Laurent, 1996 and so on)
- KL is a functional of two densities
- Very little work on nonparametric divergence estimation, especially for high-dimensional data
- Little existing work on estimating density ratio per se

Main Idea

- Variational representation of f-divergences:

  Lemma 4. *Letting $\mathcal{F}$ be any function class in $\mathcal{X} \to \mathbb{R}$, there holds:*

  $$D_\phi(\mathbb{P}, \mathbb{Q}) \geq \sup_{f \in \mathcal{F}} \int f \, d\mathbb{Q} - \phi^*(f) \, d\mathbb{P},$$

  with equality if $\mathcal{F} \cap \partial\phi(q_0/p_0) \neq \emptyset$.

  ϕ^* denotes the conjugate dual of ϕ

- Implications:

  - obtain an M-estimation procedure for divergence functional
  - also obtain the likelihood ratio function $d\mathbb{P}/d\mathbb{Q}$
  - how to choose $\mathcal{F}$
  - how to implement the optimization efficiently
  - convergence rate?

Kullback-Leibler Divergence

- For the Kullback-Leibler divergence:

$$D_K(\mathbb{P}, \mathbb{Q}) = \sup_{g>0} \int \log g \; d\mathbb{P} - \int g d\mathbb{Q} + 1.$$

- Furthermore, the supremum is attained at $g = p_0/q_0$.

M-Estimation Procedure

- Let $\mathcal{G}$ be a function class: $\mathcal{X} \to \mathbb{R}_+$

- $\int \, d\mathbb{P}_n$ and $\int \, d\mathbb{Q}_n$ denote the expectation under empirical measures $\mathbb{P}_n$ and $\mathbb{Q}_n$, respectively

- One possible estimator has the following form:

$$\hat{D}_K = \sup_{g \in \mathcal{G}} \int \log g \; d\mathbb{P}_n - \int g d\mathbb{Q}_n + 1.$$

- Supremum is attained at $\hat{g}_n$, which estimates the likelihood ratio p_0/q_0

Convex Empirical Risk with Penalty

- In practice, control the size of the function class $\mathcal{G}$ by using a penalty

- Let $I(g)$ be a measure of complexity for g

- Decompose $\mathcal{G}$ as follows:

$$\mathcal{G} = \cup_{1 \leq M \leq \infty} \mathcal{G}_M,$$

  where $\mathcal{G}_M$ is restricted to g for which $I(g) \leq M$.

- The estimation procedure involves solving:

$$\hat{g}_n = \text{argmin}_{g \in \mathcal{G}} \int g d\mathbb{Q}_n - \int \log g \; d\mathbb{P}_n + \frac{\lambda_n}{2} I^2(g).$$

Convergence Rates

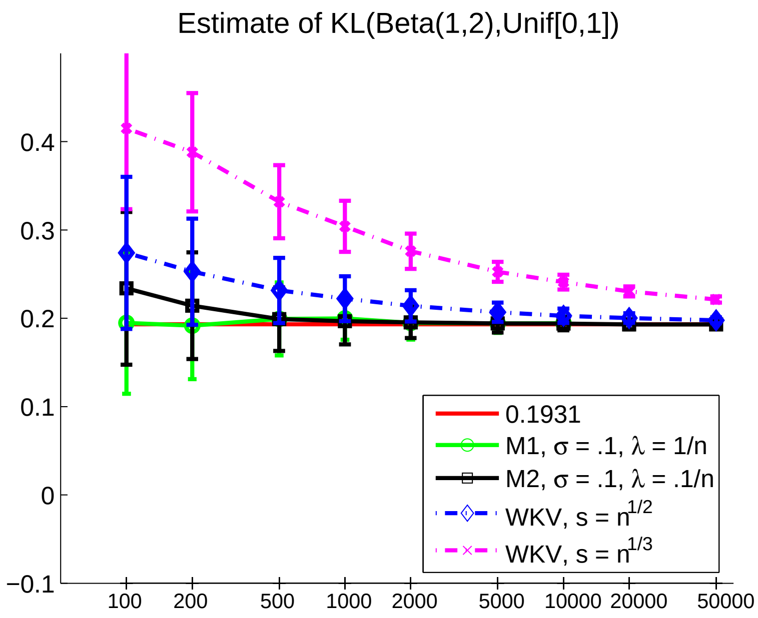
Theorem 5. *When λ_n vanishes sufficiently slowly:*

$$\lambda_n^{-1} = O_P(n^{2/(2+\gamma)})(1 + I(g_0)),$$

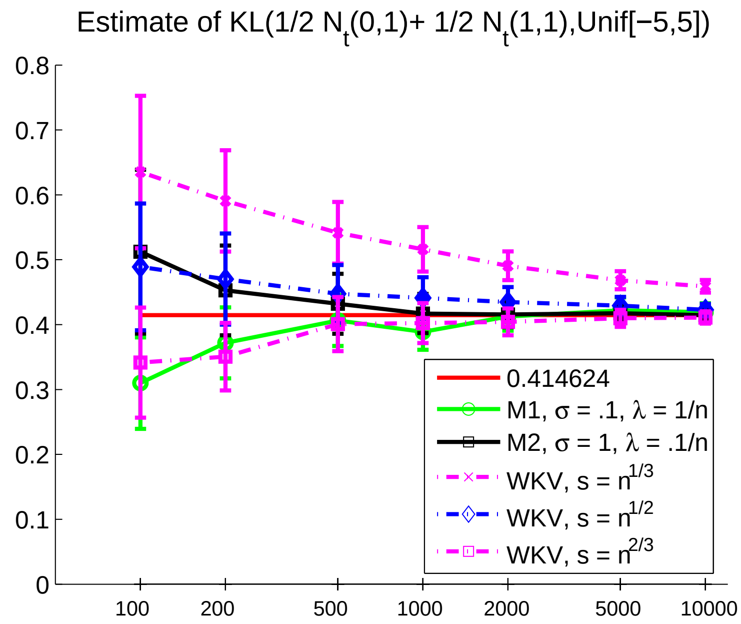
then under $\mathbb{P}$:

$$h_{\mathbb{Q}}(g_0, \hat{g}_n) = O_P(\lambda_n^{1/2})(1 + I(g_0))$$
$$I(\hat{g}_n) = O_P(1 + I(g_0)).$$

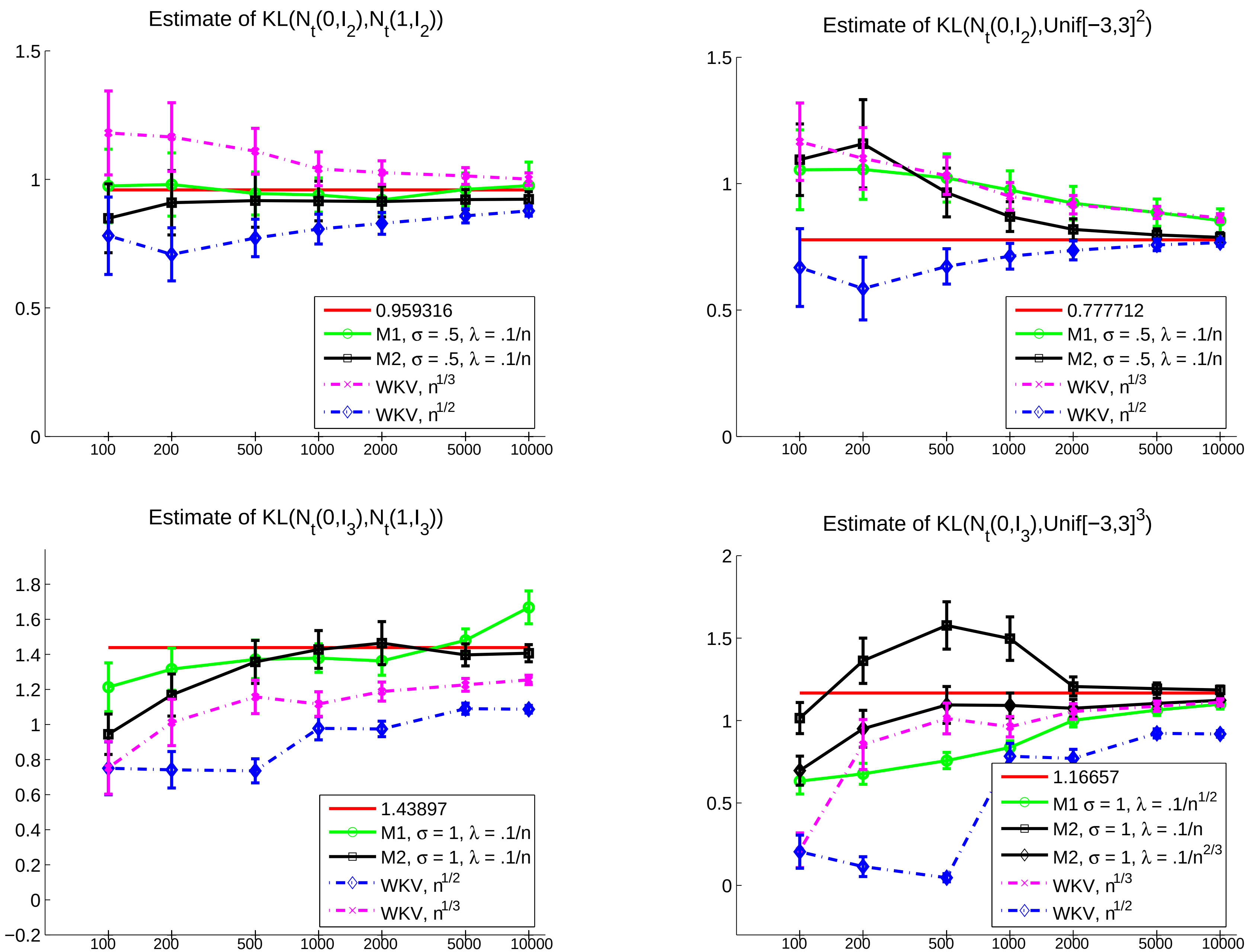
Results



Estimate of KL(Beta(1,2),Unif[0,1])
0.4
0.3
0.2
0.1
0
−0.1
100 200 500 1000 2000 5000 10000 20000 50000
0.1931
M1, σ = .1, λ = 1/n
M2, σ = .1, λ = .1/n
WKV, s = n^1/2
WKV, s = n^1/3



Estimate of KL(1/2 N_t(0,1)+ 1/2 N_t(1,1),Unif[−5,5])
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
100 200 500 1000 2000 5000 10000
0.414624
M1, σ = .1, λ = 1/n
M2, σ = 1, λ = .1/n
WKV, s = n^1/3
WKV, s = n^1/2
WKV, s = n^2/3



Estimate of KL(N_t(0,I_2),N_t(1,I_2))
0.959316
M1, σ = .5, λ = .1/n
M2, σ = .5, λ = .1/n
WKV, n^{1/3}
WKV, n^{1/2}
Estimate of KL(N_t(0,I_2),Unif[−3,3]^2)
0.777712
M1, σ = .5, λ = .1/n
M2, σ = .5, λ = .1/n
WKV, n^{1/3}
WKV, n^{1/2}
Estimate of KL(N_t(0,I_3),N_t(1,I_3))
1.43897
M1, σ = 1, λ = .1/n
M2, σ = 1, λ = .1/n
WKV, n^{1/2}
WKV, n^{1/3}
Estimate of KL(N_t(0,I_3),Unif[−3,3]^3)
1.16657
M1 σ = 1, λ = .1/n^{1/2}
M2, σ = 1, λ = .1/n
M2, σ = 1, λ = .1/n^{2/3}
WKV, n^{1/3}
WKV, n^{1/2}
100
200
500
1000
2000
5000
10000

Conclusions

- Formulated a precise link between f-divergences and surrogate loss functions
- Decision-theoretic perspective on f-divergences
- Equivalent classes of loss functions
- Can design new convex surrogate loss functions that are equivalent (in a deep sense) to 0-1 loss
  - Applications to the Bayes consistency of procedures that jointly choose an experimental design and a classifier
  - Applications to the estimation of divergences and entropy