

Lecture 5: Gaussian Process Covariance Functions

4F13: Machine Learning

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<http://mlg.eng.cam.ac.uk/teaching/4f13/>

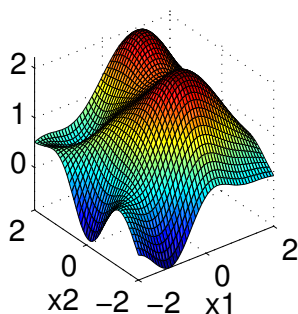
Model Selection in Practice; Hyperparameters

There are two types of task: *form* and *parameters* of the covariance function.

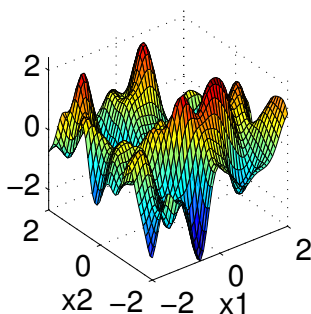
Typically, our prior is too weak to quantify aspects of the covariance function. We use a **hierarchical model** using **hyperparameters**. Eg, in ARD:

$$k(\mathbf{x}, \mathbf{x}') = v_0^2 \exp \left(- \sum_{d=1}^D \frac{(x_d - x'_d)^2}{2v_d^2} \right), \quad \text{hyperparameters } \theta = (v_0, v_1, \dots, v_d, \sigma_n^2).$$

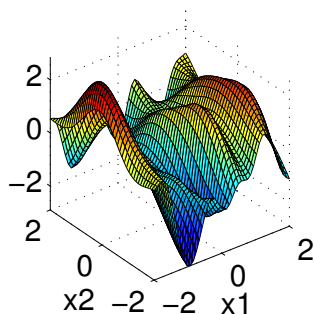
$v_1=v_2=1$



$v_1=v_2=0.32$



$v_1=0.32$ and $v_2=1$



Rational quadratic covariance function

The *rational quadratic* (RQ) covariance function:

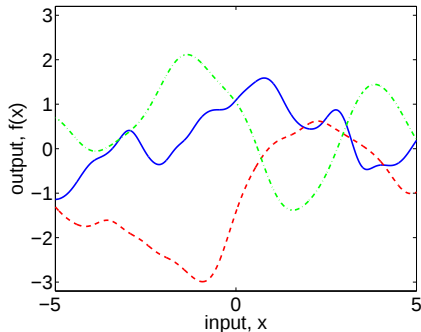
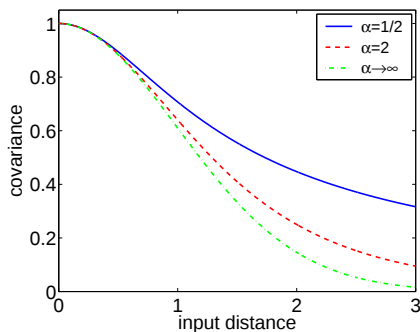
$$k_{\text{RQ}}(\mathbf{r}) = \left(1 + \frac{\mathbf{r}^2}{2\alpha\ell^2}\right)^{-\alpha}$$

with $\alpha, \ell > 0$ can be seen as a *scale mixture* (an infinite sum) of squared exponential (SE) covariance functions with different characteristic length-scales.

Using $\tau = \ell^{-2}$ and $p(\tau|\alpha, \beta) \propto \tau^{\alpha-1} \exp(-\alpha\tau/\beta)$:

$$\begin{aligned} k_{\text{RQ}}(\mathbf{r}) &= \int p(\tau|\alpha, \beta) k_{\text{SE}}(\mathbf{r}|\tau) d\tau \\ &\propto \int \tau^{\alpha-1} \exp\left(-\frac{\alpha\tau}{\beta}\right) \exp\left(-\frac{\tau\mathbf{r}^2}{2}\right) d\tau \propto \left(1 + \frac{\mathbf{r}^2}{2\alpha\ell^2}\right)^{-\alpha}, \end{aligned}$$

Rational quadratic covariance function II



The limit $\alpha \rightarrow \infty$ of the RQ covariance function is the SE.

Matérn covariance functions

Stationary covariance functions can be based on the Matérn form:

$$k(\mathbf{x}, \mathbf{x}') = \frac{1}{\Gamma(\nu)2^{\nu-1}} \left[\frac{\sqrt{2\nu}}{\ell} |\mathbf{x} - \mathbf{x}'| \right]^\nu K_\nu \left(\frac{\sqrt{2\nu}}{\ell} |\mathbf{x} - \mathbf{x}'| \right),$$

where K_ν is the modified Bessel function of second kind of order ν , and ℓ is the characteristic length scale.

Sample functions from Matérn forms are $\lfloor \nu - 1 \rfloor$ times differentiable. Thus, the hyperparameter ν can control the degree of smoothness

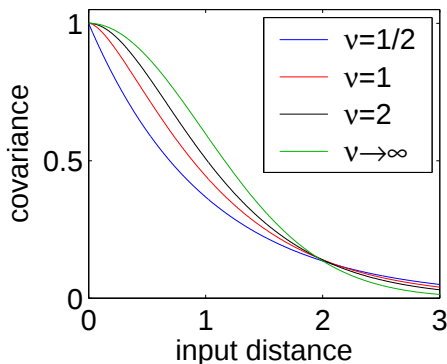
Special cases:

- $k_{\nu=1/2}(r) = \exp(-\frac{r}{\ell})$: Laplacian covariance function, Brownian motion (Ornstein-Uhlenbeck)
- $k_{\nu=3/2}(r) = (1 + \frac{\sqrt{3}r}{\ell}) \exp(-\frac{\sqrt{3}r}{\ell})$ (once differentiable)
- $k_{\nu=5/2}(r) = (1 + \frac{\sqrt{5}r}{\ell} + \frac{5r^2}{3\ell^2}) \exp(-\frac{\sqrt{5}r}{\ell})$ (twice differentiable)
- $k_{\nu \rightarrow \infty} = \exp(-\frac{r^2}{2\ell^2})$: smooth (infinitely differentiable)

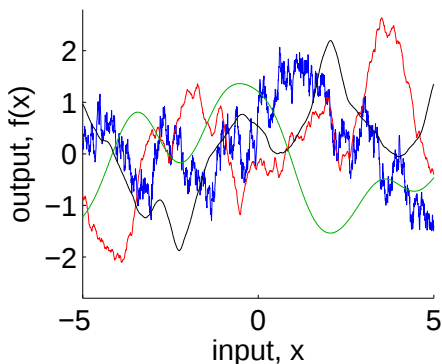
Matérn covariance functions II

Univariate Matérn covariance function with unit characteristic length scale and unit variance:

covariance function



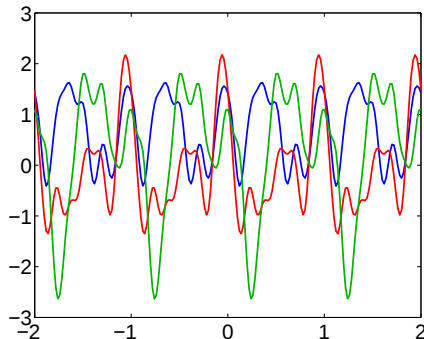
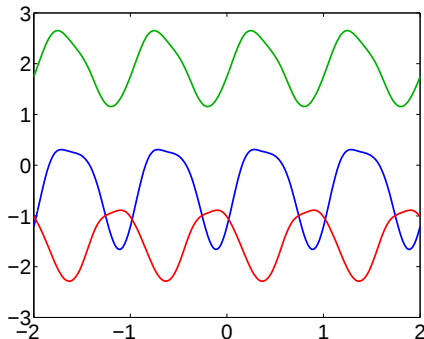
sample functions



Periodic, smooth functions

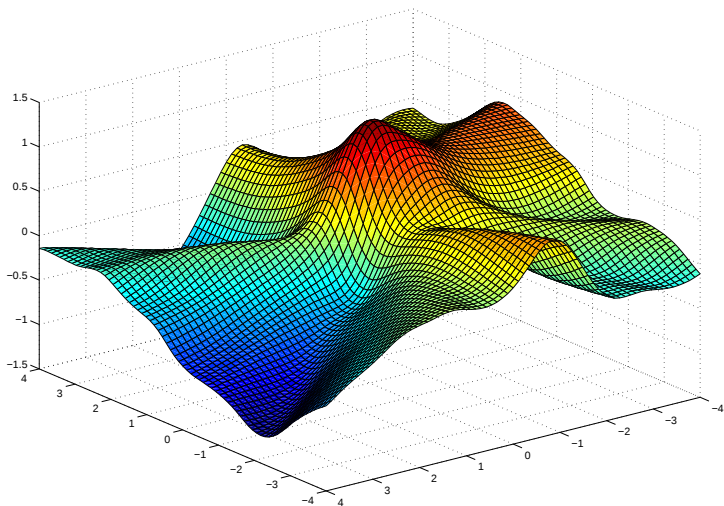
To create a distribution over periodic functions of x , we can first map the inputs to $u = (\sin(x), \cos(x))^T$, and then measure distances in the u space. Combined with the SE covariance function, which characteristic length scale ℓ , we get:

$$k_{\text{periodic}}(x, x') = \exp(-2 \sin^2(\pi(x - x'))/\ell^2)$$



Three functions drawn at random; left $\ell > 1$, and right $\ell < 1$.

Function drawn at random from a Neural Network covariance function



$$k(\mathbf{x}, \mathbf{x}') = \frac{2}{\pi} \arcsin \left(\frac{2\mathbf{x}^\top \Sigma \mathbf{x}'}{\sqrt{(1 + \mathbf{x}^\top \Sigma \mathbf{x})(1 + 2\mathbf{x}'^\top \Sigma \mathbf{x}')}} \right).$$