

Lecture 8 and 9: Message passing on Factor Graphs

4F13: Machine Learning

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<http://mlg.eng.cam.ac.uk/teaching/4f13/>

Factor Graphs

Factor graphs allow to represent the product structure of a function.

They are bipartite graphs with two types of nodes:

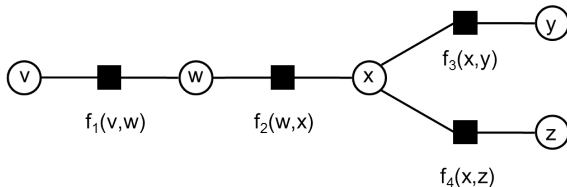
- Factor node: ■ Variable node: ○
- Edges represent the dependency of factors on variables.

Example: consider the factorising probability density function

$$p(v, w, x, y, z) = f_1(v, w)f_2(w, x)f_3(x, y)f_4(x, z)$$

- What are the marginal distributions of the individual variables?
- What is $p(w)$?

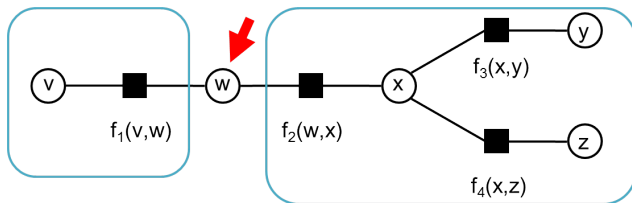
Factor trees: separation (1)



$$p(w) = \sum_v \sum_x \sum_y \sum_z f_1(v, w) f_2(w, x) f_3(x, y) f_4(x, z)$$

- If v , x , y and z take K values each, we have $\approx 3K^4$ products and $\approx K^4$ sums.
- Multiplication is distributive: $c(a + b) = ca + cb$.
The left hand is more efficient!

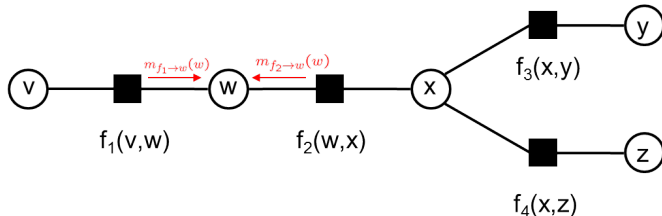
Factor trees: separation (2)



$$p(w) = \left[\sum_v f_1(v, w) \right] \cdot \left[\sum_x \sum_y \sum_z f_2(w, x) f_3(x, y) f_4(x, z) \right]$$

- From sums of products to products of sums.
- The complexity is now $\approx 2K^3$.

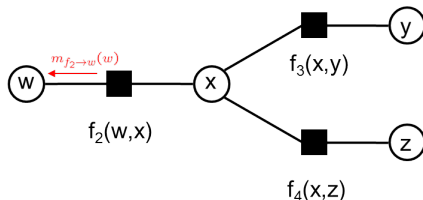
Factor trees: separation (3)



$$p(w) = \underbrace{\left[\sum_v f_1(v, w) \right]}_{m_{f_1 \rightarrow w}(w)} \cdot \underbrace{\left[\sum_x \sum_y \sum_z f_2(w, x) f_3(x, y) f_4(x, z) \right]}_{m_{f_2 \rightarrow w}(w)}$$

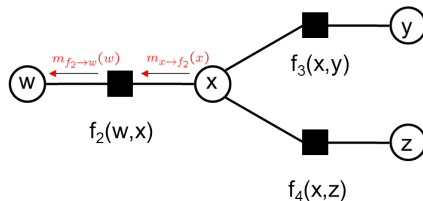
- Sums of products becomes products of sums of all messages from neighbouring factors to variable.

Messages: from factors to variables (1)



$$m_{f_2 \rightarrow w}(w) = \sum_x \sum_y \sum_z f_2(w, x) f_3(x, y) f_4(x, z)$$

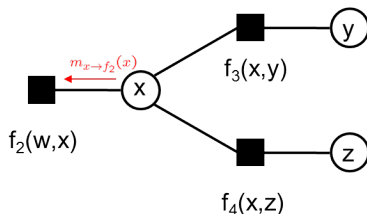
Messages: from factors to variables (2)



$$m_{f_2 \rightarrow w}(w) = \sum_x f_2(w, x) \cdot \underbrace{\left[\sum_y \sum_z f_3(x, y) f_4(x, z) \right]}_{m_{x \rightarrow f_2}(x)}$$

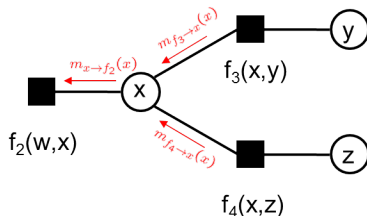
- Factors only need to sum out all their local variables.

Messages: from variables to factors (1)



$$m_{x \rightarrow f_2}(x) = \sum_y \sum_z f_3(x, y) f_4(x, z)$$

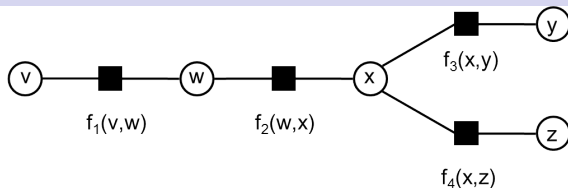
Messages: from variables to factors (2)



$$m_{x \rightarrow f_2}(x) = \underbrace{\left[\sum_y f_3(x, y) \right]}_{m_{f_3 \rightarrow x}(x)} \cdot \underbrace{\left[\sum_z f_4(x, z) \right]}_{m_{f_4 \rightarrow x}(x)}$$

- Variables pass on the product of all incoming messages.

Factor graph marginalisation: summary



$$\begin{aligned} p(w) &= \sum_v \sum_x \sum_y \sum_z f_1(v, w) f_2(w, x) f_3(x, y) f_4(x, z) \\ &= \underbrace{\left[\sum_v f_1(v, w) \right]}_{m_{f_1 \rightarrow w}(w)} \cdot \underbrace{\left[\sum_x f_2(w, x) \cdot \left[\underbrace{\left[\sum_y f_3(x, y) \right]}_{m_{f_3 \rightarrow x}(x)} \cdot \underbrace{\left[\sum_z f_4(x, z) \right]}_{m_{f_4 \rightarrow x}(x)} \right] \right]}_{m_{x \rightarrow f_2}(x)} \\ &\quad \underbrace{\hspace{10em}}_{m_{f_2 \rightarrow w}(w)} \end{aligned}$$

- The complexity is now reduced to $\approx K$.

The sum-product algorithm

Three update equations:

- Marginals are the product of all incoming messages from neighbour factors

$$p(t) = \prod_{f \in F_t} m_{f \rightarrow t}(t)$$

- Messages from factors sum out all variables except the receiving one

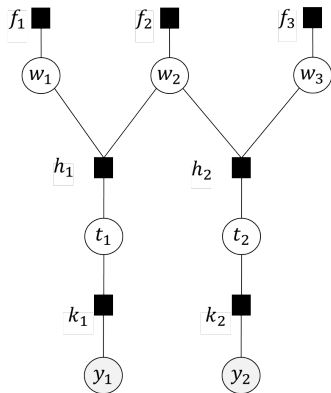
$$m_{f \rightarrow t}(t_1) = \sum_{t_2} \sum_{t_3} \dots \sum_{t_n} f(t_1, t_2, \dots, t_n) \prod_{i>1} m_{t_i \rightarrow f}(t_i)$$

- Messages from variables are the product of all incoming messages except that from the receiving factor

$$m_{t \rightarrow f}(t) = \prod_{f_j \in F_t \setminus \{f\}} m_{f_j \rightarrow t}(t)$$

Messages are results of partial computations. Computations are localised.

The full TrueSkill graph



Prior: $f_i(w_i) = \mathcal{N}(w_i; \mu_0, \sigma_0^2)$

“Game” factor:

$$h_g(w_{I_g}, w_{J_g}, t_g) = \mathcal{N}(t_g; w_{I_g} - w_{J_g}, 1)$$

(I_g and J_g are the players in game g)

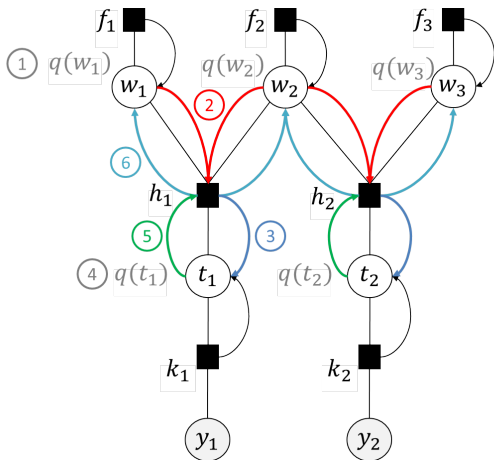
Outcome factor:

$$k_g(t_g, y_g) = \delta(y_g - \text{sign}(t_g))$$

We are interested in the marginal distributions of the skills w_i .

- What shape do these distributions have?
- We need to make some approximations.
- We will also pretend the structure is a tree (ignore loops).

Expectation Propagation in the full TrueSkill graph



Iterate

- (1) Update skill marginals.
- (2) Compute skill to game messages.
- (3) Compute game to performance messages.
- (4) Approximate performance marginals.
- (5) Compute performance to game messages.
- (6) Compute game to skill messages.

Message passing for TrueSkill

$$m_{h_g \rightarrow w_{I_g}}^{\tau=0}(w_{I_g}) = 1, \quad m_{h_g \rightarrow w_{J_g}}^{\tau=0}(w_{J_g}) = 1, \quad \forall g,$$

$$q^\tau(w_i) = f(w_i) \prod_{g=1}^N m_{h_g \rightarrow w_i}^\tau(w_i) \sim \mathcal{N}(\mu_i, \sigma_i^2),$$

$$m_{w_{I_g} \rightarrow h_g}^\tau(w_{I_g}) = \frac{q^\tau(w_{I_g})}{m_{h_g \rightarrow w_{I_g}}^\tau(w_{I_g})}, \quad m_{w_{J_g} \rightarrow h_g}^\tau(w_{J_g}) = \frac{q^\tau(w_{J_g})}{m_{h_g \rightarrow w_{J_g}}^\tau(w_{J_g})},$$

$$m_{h_g \rightarrow t_g}^\tau(t_g) = \iint h_g(t_g, w_{I_g}, w_{J_g}) m_{w_{I_g} \rightarrow h_g}^\tau(w_{I_g}) m_{w_{J_g} \rightarrow h_g}^\tau(w_{J_g}) dw_{I_g} dw_{J_g},$$

$$q^{\tau+1}(t_g) = \text{Approx}(m_{h_g \rightarrow t_g}^\tau(t_g) m_{k_g \rightarrow t_g}(t_g)),$$

$$m_{t_g \rightarrow h_g}^{\tau+1}(t_g) = \frac{q^{\tau+1}(t_g)}{m_{h_g \rightarrow t_g}^\tau(t_g)},$$

$$m_{h_g \rightarrow w_{I_g}}^{\tau+1}(w_{I_g}) = \iint h_g(t_g, w_{I_g}, w_{J_g}) m_{t_g \rightarrow h_g}^{\tau+1}(t_g) m_{w_{J_g} \rightarrow h_g}^\tau(w_{J_g}) dt_g dw_{J_g},$$

$$m_{h_g \rightarrow w_{J_g}}^{\tau+1}(w_{J_g}) = \iint h_g(t_g, w_{J_g}, w_{I_g}) m_{t_g \rightarrow h_g}^{\tau+1}(t_g) m_{w_{I_g} \rightarrow h_g}^\tau(w_{I_g}) dt_g dw_{I_g}.$$

In a little more detail

At iteration τ messages m and marginals q are Gaussian, with *means* μ , *standard deviations* σ , *variances* $v = \sigma^2$, *precisions* $r = v^{-1}$ and *natural means* $\lambda = r\mu$.

Step 0 Initialise incoming skill messages:

$$\left. \begin{aligned} r_{h_g \rightarrow w_i}^{\tau=0} &= 0 \\ \mu_{h_g \rightarrow w_i}^{\tau=0} &= 0 \end{aligned} \right\} m_{h_g \rightarrow w_i}^{\tau=0}(w_i)$$

Step 1 Compute marginal skills:

$$\left. \begin{aligned} r_i^\tau &= r_0 + \sum_g r_{h_g \rightarrow w_i}^\tau \\ \lambda_i^\tau &= \lambda_0 + \sum_g \lambda_{h_g \rightarrow w_i}^\tau \end{aligned} \right\} q^\tau(w_i)$$

Step 2 Compute skill to game messages:

$$\left. \begin{aligned} r_{w_i \rightarrow h_g}^\tau &= r_i^\tau - r_{h_g \rightarrow w_i}^\tau \\ \lambda_{w_i \rightarrow h_g}^\tau &= \lambda_i^\tau - \lambda_{h_g \rightarrow w_i}^\tau \end{aligned} \right\} m_{w_i \rightarrow h_g}^\tau(w_i)$$

Step 3 Game to performance messages:

$$\left. \begin{aligned} v_{h_g \rightarrow t_g}^\tau &= 1 + v_{w_{I_g} \rightarrow h_g}^\tau + v_{w_{J_g} \rightarrow h_g}^\tau \\ \mu_{h_g \rightarrow t_g}^\tau &= \mu_{I_g \rightarrow h_g}^\tau - \mu_{J_g \rightarrow h_g}^\tau \end{aligned} \right\} m_{h_g \rightarrow t_g}^\tau(t_g)$$

Step 4 Compute marginal performances:

$$\begin{aligned} p(t_g) &\propto \mathcal{N}(\mu_{h_g \rightarrow t_g}^\tau, v_{h_g \rightarrow t_g}^\tau) \mathbb{I}(y - \text{sign}(t)) \\ &\simeq \mathcal{N}(\tilde{\mu}_g^{\tau+1}, \tilde{v}_g^{\tau+1}) = q^{\tau+1}(t_g) \end{aligned}$$

We find the parameters of q by *moment matching*

$$\left. \begin{aligned} \tilde{v}_g^{\tau+1} &= v_{h_g \rightarrow t_g}^\tau \left(1 - \Lambda\left(\frac{\mu_{h_g \rightarrow t_g}^\tau}{\sigma_{h_g \rightarrow t_g}^\tau}\right) \right) \\ \tilde{\mu}_g^{\tau+1} &= \mu_{h_g \rightarrow t_g}^\tau + \sigma_{h_g \rightarrow t_g}^\tau \Psi\left(\frac{\mu_{h_g \rightarrow t_g}^\tau}{\sigma_{h_g \rightarrow t_g}^\tau}\right) \end{aligned} \right\} q^{\tau+1}(t_g)$$

where we have defined $\Psi(x) = \mathcal{N}(x)/\Phi(x)$ and $\Lambda(x) = \Psi(x)(\Psi(x) + x)$.

Step 5 Performance to game message:

$$\left. \begin{aligned} r_{t_g \rightarrow h_g}^{\tau+1} &= \tilde{r}_g^{\tau+1} - r_{h_g \rightarrow t_g}^{\tau} \\ \lambda_{t_g \rightarrow h_g}^{\tau+1} &= \tilde{\lambda}_g^{\tau+1} - \lambda_{h_g \rightarrow t_g}^{\tau} \end{aligned} \right\} m_{t_g \rightarrow h_g}^{\tau+1}(t_g)$$

Step 6 Game to skill message:

For player 1 (the winner):

$$\left. \begin{aligned} v_{h_g \rightarrow w_{I_g}}^{\tau+1} &= 1 + v_{t_g \rightarrow h_g}^{\tau+1} + v_{w_{J_g} \rightarrow h_g}^{\tau} \\ \mu_{h_g \rightarrow w_{I_g}}^{\tau+1} &= \mu_{w_{J_g} \rightarrow h_g}^{\tau} + \mu_{t_g \rightarrow h_g}^{\tau+1} \end{aligned} \right\} m_{h_g \rightarrow w_{I_g}}^{\tau+1}(w_{I_g})$$

and for player 2 (the loser):

$$\left. \begin{aligned} v_{h_g \rightarrow w_{J_g}}^{\tau+1} &= 1 + v_{t_g \rightarrow h_g}^{\tau+1} + v_{w_{I_g} \rightarrow h_g}^{\tau} \\ \mu_{h_g \rightarrow w_{J_g}}^{\tau+1} &= \mu_{w_{I_g} \rightarrow h_g}^{\tau} - \mu_{t_g \rightarrow h_g}^{\tau+1} \end{aligned} \right\} m_{h_g \rightarrow w_{J_g}}^{\tau+1}(w_{J_g})$$

Go back to **Step 1** with $\tau := \tau + 1$ (or stop).

Moments of a truncated Gaussian density (1)

Consider the truncated Gaussian density function

$$p(t) = \frac{1}{Z_t} \delta(y - \text{sign}(t)) \mathcal{N}(t; \mu, \sigma^2)$$

where $y \in \{-1, 1\}$ and $\delta(x) = 1$ only if $x = 0$ ($\delta(x) = 0$ if $x \neq 0$).

We have seen that the normalisation constant is $Z_t = \Phi\left(\frac{y\mu}{\sigma}\right)$.

We want to approximate $p(t)$ by a Gaussian density function $q(t)$ with mean and variance equal to the first and second central moments of $p(t)$.

This means we need to compute:

- First moment: $\mathbb{E}[t] = \langle t \rangle_{p(t)}$
- Second central moment: $\mathbb{V}[t] = \langle t^2 \rangle_{p(t)} - \langle t \rangle_{p(t)}^2$

Moments of a truncated Gaussian density (2)

First moment. We take the derivative of Z_t wrt. μ :

$$\begin{aligned}\frac{\partial Z_t}{\partial \mu} &= \frac{\partial}{\partial \mu} \int_0^{+\infty} N(t; y\mu, \sigma^2) dt = \int_0^{+\infty} \frac{\partial}{\partial \mu} N(t; y\mu, \sigma^2) dt \\ &= \int_0^{+\infty} y\sigma^{-2}(t - y\mu)N(t; y\mu, \sigma^2) dt = yZ_t\sigma^{-2} \int_{-\infty}^{+\infty} (t - y\mu)p(t) dt \\ &= yZ_t\sigma^{-2} \langle t - y\mu \rangle_{p(t)} = yZ_t\sigma^{-2} \langle t \rangle_{p(t)} - \mu Z_t \sigma^{-2}\end{aligned}$$

where $\langle t \rangle_{p(t)}$ is the expectation of t under $p(t)$. We can also write:

$$\frac{\partial Z_t}{\partial \mu} = \frac{\partial}{\partial \mu} \Phi\left(\frac{y\mu}{\sigma}\right) = yN(y\mu; 0, \sigma^2)$$

Combining both expressions for $\frac{\partial Z_t}{\partial \mu}$ we obtain

$$\langle t \rangle_{p(t)} = y\mu + \sigma^2 \frac{N(y\mu; 0, \sigma^2)}{\Phi(\frac{y\mu}{\sigma})} = y\mu + \sigma \frac{N(\frac{y\mu}{\sigma}; 0, 1)}{\Phi(\frac{y\mu}{\sigma})} = y\mu + \sigma \Psi\left(\frac{y\mu}{\sigma}\right)$$

where use $N(y\mu; 0, \sigma^2) = \sigma^{-1}N(\frac{y\mu}{\sigma}; 0, 1)$ and define $\Psi(z) = \frac{N(z; 0, 1)}{\Phi(z)}$.

Moments of a truncated Gaussian density (3)

Second moment. We take the second derivative of Z_t wrt. μ :

$$\begin{aligned}\frac{\partial^2 Z_t}{\partial \mu^2} &= \frac{\partial}{\partial \mu} \int_0^{+\infty} y \sigma^{-2} (t - y\mu) \mathcal{N}(t; y\mu, \sigma^2) dt \\ &= \Phi\left(\frac{y\mu}{\sigma}\right) \langle -\sigma^{-2} + \sigma^{-4} (t - y\mu)^2 \rangle_{p(t)}\end{aligned}$$

We can also write

$$\frac{\partial^2 Z_t}{\partial \mu^2} = \frac{\partial}{\partial \mu} y \mathcal{N}(y\mu; 0, \sigma^2) = -\sigma^{-2} y \mu \mathcal{N}(y\mu; 0, \sigma^2)$$

Combining both we obtain

$$\mathbb{V}[t] = \sigma^2 \left(1 - \Lambda\left(\frac{y\mu}{\sigma}\right)\right)$$

where we define $\Lambda(z) = \Psi(z)(\Psi(z) + z)$.