Some useful Gaussian and matrix equations

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Appendix: Some useful Gaussian identities

If \( x \) is multivariate Gaussian with mean \( \mu \) and covariance matrix \( \Sigma \)

\[
p(x; \mu, \Sigma) = (2\pi|\Sigma|)^{-D/2} \exp \left( -\frac{(x - \mu)^\top \Sigma^{-1} (x - \mu)}{2} \right),
\]

then

\[
\mathbb{E}[x] = \mu,
\]
\[
\mathbb{V}[x] = \mathbb{E}[(x - \mathbb{E}[x])^2] = \Sigma.
\]

For any matrix \( A \), if \( z = Ax \) then \( z \) is Gaussian and

\[
\mathbb{E}[z] = A\mu,
\]
\[
\mathbb{V}[z] = A\Sigma A^\top.
\]
Matrix and Gaussian identities cheat sheet

Matrix identities

• Matrix inversion lemma (Woodbury, Sherman & Morrison formula)

\[(Z + UWV^\top)^{-1} = Z^{-1} - Z^{-1}U(W^{-1} + V^\top Z^{-1}U)^{-1}V^\top Z^{-1}\]

• A similar equation exists for determinants

\[|Z + UWV^\top| = |Z| |W| |W^{-1} + V^\top Z^{-1}U|\]

The product of two Gaussian density functions

\[\mathcal{N}(x|a, A) \mathcal{N}(P^\top x|b, B) = z_c \mathcal{N}(x|c, C)\]

• is proportional to a Gaussian density function with covariance and mean

\[C = (A^{-1} + PB^{-1}P^\top)^{-1} \quad c = C (A^{-1}a + PB^{-1}b)\]

• and has a normalizing constant \(z_c\) that is Gaussian both in \(a\) and in \(b\)

\[z_c = (2 \pi)^{-\frac{m}{2}} |B + P^\top A P|^{-\frac{1}{2}} \exp \left(-\frac{1}{2}(b - P^\top a)^\top (B + P^\top A P)^{-1} (b - P^\top a)\right)\]