Finite and infinite basis GPs

Carl Edward Rasmussen

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Key concepts

• Should we use finite or infinite models?
• GPs are a fancy way of using infinite models, but
  • will it actually make any difference in practise?
• finite models correspond to much stronger assumptions about the data
• therefore, we don’t want to use finite models
• a GP with squared exponential covariance function corresponds to an infinite linear in the parameters model with Gaussian bumps everywhere
• illustrate the difference

Cromwell’s dictum

*I beseech you, in the bowels of Christ, consider it possible that you are mistaken*  
— Oliver Cromwell, 1650
From infinite linear models to Gaussian processes

Consider the class of functions (sums of squared exponentials):

\[ f(x) = \lim_{N \to \infty} \frac{1}{N} \sum_{n=-N/2}^{N/2} \gamma_n \exp\left(-\left(x - \frac{n}{\sqrt{N}}\right)^2\right), \quad \text{where } \gamma_n \sim \mathcal{N}(0, 1), \quad \forall n \]

\[ = \int_{-\infty}^{\infty} \gamma(u) \exp\left(-\left(x - u\right)^2\right) du, \quad \text{where } \gamma(u) \sim \mathcal{N}(0, 1), \quad \forall u. \]

The mean function is:

\[ \mu(x) = \mathbb{E}[f(x)] = \int_{-\infty}^{\infty} \exp\left(-\left(x - u\right)^2\right) \int_{-\infty}^{\infty} \gamma(u) p(\gamma(u)) d\gamma(u) \ du = 0, \]

and the covariance function:

\[ \mathbb{E}[f(x)f(x')] = \int \exp\left(- \left(x - u\right)^2 - \left(x' - u\right)^2\right) du \]

\[ = \int \exp\left(- 2\left(u - \frac{x + x'}{2}\right)^2 + \frac{(x + x')^2}{2} - x^2 - x'^2\right) du \propto \exp\left(-\frac{(x - x')^2}{2}\right). \]

Thus, the squared exponential covariance function is equivalent to regression using infinitely many Gaussian shaped basis functions placed everywhere, not just at your training points!

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Using finitely many basis functions may be dangerous!(1)

Finite linear model with 5 localized basis functions

Gaussian process with infinitely many localized basis functions
Using finitely many basis functions may be dangerous!(2)

Finite linear model with 5 localized basis functions:

Gaussian process with infinitely many localized basis functions:
Using finitely many basis functions may be dangerous!(3)

Finite linear model with 5 localized basis functions

Gaussian process with infinitely many localized basis functions