Discrete Binary Distributions

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Key concepts

- Bernoulli: probabilities over binary variables
- Binomial: probabilities over counts and binary sequences
- Inference, priors and pseudo-counts, the Beta distribution
- model comparison: an example

Coin tossing

- You are presented with a coin: what is the probability of heads?

  What does this question even mean?

- How much are you willing to bet $p(\text{head}) > 0.5$?

  Do you expect this coin to come up heads more often that tails?

  Wait... can you toss the coin a few times, I need data!

- Ok, you observe the following sequence of outcomes (T: tail, H: head):

  H

  This is not enough data!

- Now you observe the outcome of three additional tosses:

  HHTH

  How much are you *now* willing to bet $p(\text{head}) > 0.5$?

The Bernoulli discrete binary distribution

The *Bernoulli* probability distribution over binary random variables:

- Binary random variable X: outcome x of a single coin toss.
- The two values x can take are
  - $X = 0$ for tail,
  - $X = 1$ for heads.
- Let the probability of heads be $\pi = p(X = 1)$.
  π is the *parameter* of the Bernoulli distribution.
- The probability of tail is $p(X = 0) = 1 - \pi$. We can compactly write

$$p(X = x|\pi) \;=\; p(x|\pi) \;=\; \pi^x (1-\pi)^{1-x}$$

What do we think π is after observing a single heads outcome?

- Maximum likelihood! Maximise $p(H|\pi)$ with respect to π:

$$p(H|\pi) \;=\; p(x = 1|\pi) \;=\; \pi, \qquad \operatorname{argmax}_{\pi \in [0,1]} \pi = 1$$

- Ok, so the answer is $\pi = 1$. This coin only generates heads.

Is this reasonable? How much are you willing to bet p(heads)>0.5?

The binomial distribution: counts of binary outcomes

We observe a sequence of tosses rather than a single toss:

$$\mathsf{HHTH}$$

- The probability of this particular sequence is: $p(\mathsf{HHTH}) = \pi^3(1-\pi)$.
- But so is the probability of THHH, of HTHH and of HHHT.
- We often don't care about the order of the outcomes, only about the *counts*. In our example the probability of 3 heads out of 4 tosses is: $4\pi^3(1-\pi)$.

The *binomial distribution* gives the probability of observing k heads out of n tosses

$$p(k|\pi, n) \; = \; \binom{n}{k}\pi^k(1-\pi)^{n-k}$$

- This assumes n independent tosses from a Bernoulli distribution $p(x|\pi)$.
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ is the binomial coefficient, also known as "n choose k".

Maximum likelihood under a binomial distribution

If we observe k heads out of n tosses, what do we think π is?
We can maximise the likelihood of parameter π given the observed data.

$$p(k|\pi, n) \;\propto\; \pi^k (1-\pi)^{n-k}$$

It is convenient to take the logarithm and derivatives with respect to π

$$\log p(k|\pi, n) \;=\; k \log \pi + (n-k)\log(1-\pi) + \text{Constant}$$

$$\frac{\partial \log p(k|\pi, n)}{\partial \pi} \;=\; \frac{k}{\pi} - \frac{n-k}{1-\pi} = 0 \quad \Longleftrightarrow \quad \boxed{\pi \;=\; \frac{k}{n}}$$

Is this reasonable?

- For HHTH we get $\pi = 3/4$.
- How much would you bet now that $p(\text{heads}) > 0.5$?

What do you think $p(\pi > 0.5)$*is?*

Wait! This is a probability over ... a probability?

Prior beliefs about coins – before tossing the coin

So you have observed 3 heads out of 4 tosses but are unwilling to bet £100 that $p(\text{heads}) > 0.5$?

(That for example out of 10,000,000 tosses at least 5,000,001 will be heads)

Why?

- You might believe that coins tend to be fair ($\pi \simeq \frac{1}{2}$).
- A finite set of observations *updates your opinion* about π.
- But how to express your opinion about π *before* you see any data?

Pseudo-counts: You think the coin is fair and... you are...

- Not very sure. You act as if you had seen 2 heads and 2 tails before.
- Pretty sure. It is as if you had observed 20 heads and 20 tails before.
- Totally sure. As if you had seen 1000 heads and 1000 tails before.

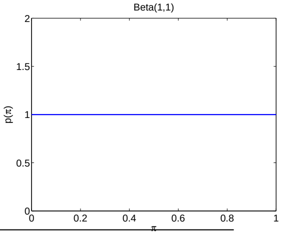
Depending on the strength of your prior assumptions, it takes a different number of actual observations to change your mind.

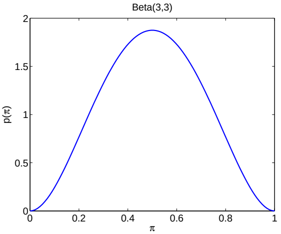
The Beta distribution: distributions on *probabilities*

Continuous probability distribution defined on the interval $[0, 1]$

$$\mathrm{Beta}(\pi|\alpha,\beta) \;=\; \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\pi^{\alpha-1}(1-\pi)^{\beta-1} \;=\; \frac{1}{\mathrm{B}(\alpha,\beta)}\pi^{\alpha-1}(1-\pi)^{\beta-1}$$

- $\alpha > 0$ and $\beta > 0$ are the shape *parameters*.
- these parameters correspond to ‘one plus the pseudo-counts’.
- $\Gamma(\alpha)$ is an extension of the factorial function[1]. $\Gamma(n) = (n-1)!$ for integer n.
- $\mathrm{B}(\alpha,\beta)$ is the beta function, it normalises the Beta distribution.
- The mean is given by $\mathrm{E}(\pi) = \frac{\alpha}{\alpha+\beta}$.   [Left: $\alpha = \beta = 1$, Right: $\alpha = \beta = 3$]





[1] $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}\,dx$

Posterior for coin tossing

Imagine we observe a single coin toss and it comes out heads. Our observed data is:

$$\mathcal{D} \;=\; \{k = 1\}, \quad \text{where} \quad n = 1.$$

The probability of the observed data given π is the *likelihood*:

$$p(\mathcal{D}|\pi) \;=\; \pi$$

We use our *prior* $p(\pi|\alpha,\beta) = \text{Beta}(\pi|\alpha,\beta)$ to get the *posterior* probability:

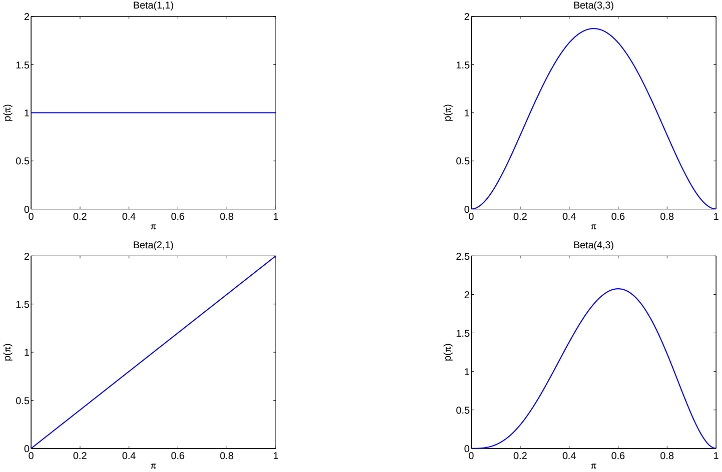
$$\begin{aligned} p(\pi|\mathcal{D}) \;&=\; \frac{p(\pi|\alpha,\beta)p(\mathcal{D}|\pi)}{p(\mathcal{D})} \;\propto\; \pi\,\text{Beta}(\pi|\alpha,\beta) \\ &\propto\; \pi\,\pi^{(\alpha-1)}(1-\pi)^{(\beta-1)} \;\propto\; \text{Beta}(\pi|\alpha+1,\beta) \end{aligned}$$

The Beta distribution is a *conjugate* prior to the Bernoulli/binomial distribution:

- The resulting posterior is also a Beta distribution.
- The posterior parameters are given by: $\begin{aligned} \alpha_{\text{posterior}} &= \alpha_{\text{prior}} + k \\ \beta_{\text{posterior}} &= \beta_{\text{prior}} + (n-k) \end{aligned}$

Before and after observing one head

Prior



Posterior

Making predictions

Given some data $\mathcal{D}$, what is the predicted probability of the next toss being heads, $x_{\text{next}} = 1$?
Under the Maximum Likelihood approach we predict using the value of π_{ML} that maximises the likelihood of π given the observed data, $\mathcal{D}$:

$$p(x_{\text{next}} = 1|\pi_{\text{ML}}) = \pi_{\text{ML}}$$

With the Bayesian approach, **average** over all possible parameter settings:

$$p(x_{\text{next}} = 1|\mathcal{D}) = \int p(x = 1|\pi)\, p(\pi|\mathcal{D})\, d\pi$$

The prediction for heads happens to correspond to the mean of the *posterior* distribution. E.g. for $\mathcal{D} = \{(x = 1)\}$:

- Learner A with $\text{Beta}(1, 1)$ predicts $p(x_{\text{next}} = 1|\mathcal{D}) = \frac{2}{3}$
- Learner B with $\text{Beta}(3, 3)$ predicts $p(x_{\text{next}} = 1|\mathcal{D}) = \frac{4}{7}$

Making predictions - other statistics

Given the posterior distribution, we can also answer other questions such as "what is the probability that $\pi > 0.5$ given the observed data?"

$$p(\pi > 0.5|\mathcal{D}) \;=\; \int_{0.5}^{1} p(\pi'|\mathcal{D})\,d\pi' \;=\; \int_{0.5}^{1} \text{Beta}(\pi'|\alpha', \beta')d\pi'$$

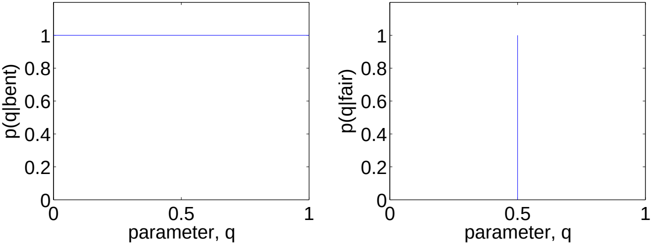
- Learner A with prior $\text{Beta}(1, 1)$ predicts $p(\pi > 0.5|\mathcal{D}) = 0.75$
- Learner B with prior $\text{Beta}(3, 3)$ predicts $p(\pi > 0.5|\mathcal{D}) = 0.66$

Learning about a coin, multiple models (1)

Consider two alternative models of a coin, "fair" and "bent". A priori, we may think that "fair" is more probable, eg:

$$p(\text{fair}) = 0.8, \qquad p(\text{bent}) = 0.2$$

For the bent coin, (a little unrealistically) all parameter values could be equally likely, where the fair coin has a fixed probability:



Learning about a coin, multiple models (2)

We make 10 tosses, and get data $\mathcal{D}$: T H T H T T T T T T
The evidence for the fair model is: $p(\mathcal{D}|\text{fair}) = (1/2)^{10} \simeq 0.001$
and for the bent model:

$$p(\mathcal{D}|\text{bent}) = \int p(\mathcal{D}|\pi, \text{bent})p(\pi|\text{bent})\ d\pi = \int \pi^2(1-\pi)^8\ d\pi = B(3,9) \simeq 0.002$$

Using priors $p(\text{fair}) = 0.8$, $p(\text{bent}) = 0.2$, the posterior by Bayes rule:

$$p(\text{fair}|\mathcal{D}) \propto 0.0008, \qquad p(\text{bent}|\mathcal{D}) \propto 0.0004,$$

ie, two thirds probability that the coin is fair.
How do we make predictions? By weighting the predictions from each model by their probability. Probability of Head at next toss is:

$$\frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{3}{12} = \frac{5}{12}.$$