Moment matching approximation

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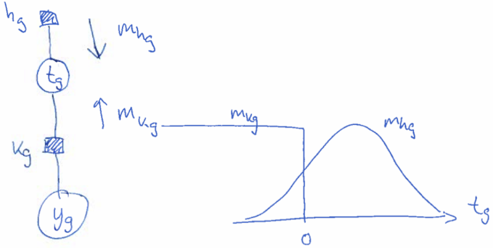
Key concepts

- in practise, we can (more or less) only compute with Gaussians
- but the game outcomes are binary
- how can we approximate a binary variable with a Gaussian?
- key idea: moment matching approximates the effect of the binary variable

Approximating a step by a Gaussian?



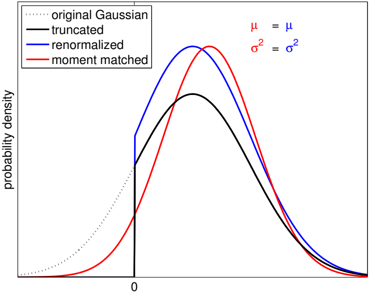
Q: How do we approximate m_{kg} by a Gaussian?

Does that even make sense?

Moments of a truncated Gaussian density (1)

Consider the truncated Gaussian density function

$$p(t) \;=\; \frac{1}{Z_t}\delta\big(y - \mathrm{sign}(t)\big)\mathcal{N}(t;\mu,\sigma^2) \;\; \text{where} \;\; y \in \{-1, 1\} \;\; \text{and} \;\; \delta(x) = 1 \;\; \text{iff} \;\; x = 0.$$



We want to *approximate* $p(t)$ by a Gaussian density function $q(t)$ with mean and variance equal to the first and second central moments of $p(t)$. We need:

- First moment: $\mathbb{E}[t] = \langle t \rangle_{p(t)}$
- Second central moment: $\mathbb{V}[t] = \langle t^2 \rangle_{p(t)} - \langle t \rangle^2_{p(t)}$

Moments of a truncated Gaussian density (2)

We have seen that the normalisation constant is $Z_t = \Phi\big(\frac{y\mu}{\sigma}\big)$.
First moment. We take the derivative of Z_t wrt. μ:

$$\begin{aligned}\frac{\partial Z_t}{\partial \mu} &= \frac{\partial}{\partial \mu}\int_0^{+\infty} N(t; y\mu, \sigma^2)dt = \int_0^{+\infty}\frac{\partial}{\partial \mu} N(t; y\mu, \sigma^2)dt \\ &= \int_0^{+\infty} y\sigma^{-2}(t - y\mu)N(t; y\mu, \sigma^2)dt = yZ_t\sigma^{-2}\int_{-\infty}^{+\infty}(t - y\mu)p(t)dt \\ &= yZ_t\sigma^{-2}\langle t - y\mu\rangle_{p(t)} = yZ_t\sigma^{-2}\langle t\rangle_{p(t)} - \mu Z_t\sigma^{-2}\end{aligned}$$

where $\langle t\rangle_{p(t)}$ is the expectation of t under $p(t)$. We can also write:

$$\frac{\partial Z_t}{\partial \mu} = \frac{\partial}{\partial \mu}\Phi\big(\frac{y\mu}{\sigma}\big) = y\mathcal{N}(y\mu; 0, \sigma^2)$$

Combining both expressions for $\frac{\partial Z_t}{\partial \mu}$ we obtain

$$\langle t\rangle_{p(t)} = y\mu + \sigma^2\frac{\mathcal{N}(y\mu; 0, \sigma^2)}{\Phi(\frac{y\mu}{\sigma})} = y\mu + \sigma\frac{\mathcal{N}(\frac{y\mu}{\sigma}; 0, 1)}{\Phi(\frac{y\mu}{\sigma})} = y\mu + \sigma\Psi\big(\frac{y\mu}{\sigma}\big)$$

where use $\mathcal{N}(y\mu; 0, \sigma^2) = \sigma^{-1}\mathcal{N}(\frac{y\mu}{\sigma}; 0, 1)$ and define $\Psi(z) = \frac{N(z;0,1)}{\Phi(z)}$.

Moments of a truncated Gaussian density (3)

Second moment. We take the second derivative of Z_t wrt. μ:

$$\frac{\partial^2 Z_t}{\partial \mu^2} = \frac{\partial}{\partial \mu}\int_0^{+\infty} y\sigma^{-2}(t - y\mu)N(t; y\mu, \sigma^2)dt$$
$$= \Phi\big(\frac{y\mu}{\sigma}\big)\langle -\sigma^{-2} + \sigma^{-4}(t - y\mu)^2\rangle_{p(t)}$$

We can also write

$$\frac{\partial^2 Z_t}{\partial \mu^2} = \frac{\partial}{\partial \mu} y\mathcal{N}(y\mu; 0, \sigma^2) = -\sigma^{-2} y\mu\mathcal{N}(y\mu; 0, \sigma^2)$$

Combining both we obtain

$$\mathbb{V}[t] = \sigma^2\big(1 - \Lambda(\frac{y\mu}{\sigma})\big)$$

where we define $\Lambda(z) = \Psi(z)\big(\Psi(z) + z\big)$.