4F13: Machine Learning

Propagation on Factor Graphs

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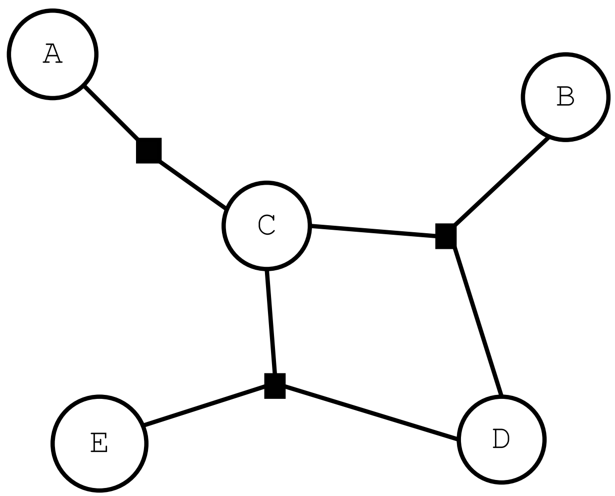
Factor Graphs

In a factor graph, the joint probability distribution is written as a product of factors. Consider a vector of variables $\mathbf{x} = (x_1, \ldots, x_n)$

$$p(\mathbf{x}) = p(x_1, \ldots, x_n) = \frac{1}{Z} \prod_j f_j(\mathbf{x}_{S_j})$$

where Z is the normalisation constant, S_j denotes the subset of $\{1, \ldots, n\}$ which participate in factor f_j and $\mathbf{x}_{S_j} = \{x_i : i \in S_j\}$.



variables nodes: we draw open circles for each variable x_i in the distribution.
function nodes: we draw filled dots for each function f_j in the distribution.

Propagation in Factor Graphs

Let $\mathrm{n}(x)$ denote the set of function nodes that are neighbors of x.
Let $\mathrm{n}(f)$ denote the set of variable nodes that are neighbors of f.

We can compute probabilities in a factor graph by propagating messages from variable nodes to function nodes and viceversa.

message from variable x to local function f:

$$\mu_{x \to f}(x) = \prod_{h \in \mathrm{n}(x) \setminus \{f\}} \mu_{h \to x}(x)$$

message from local function f to variable x:

$$\mu_{f \to x}(x) = \sum_{\mathbf{x} \setminus x} \left(f(\mathbf{x}) \prod_{y \in \mathrm{n}(f) \setminus \{x\}} \mu_{y \to f}(y) \right)$$

Propagation in Factor Graphs

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message from local function f to variable x:

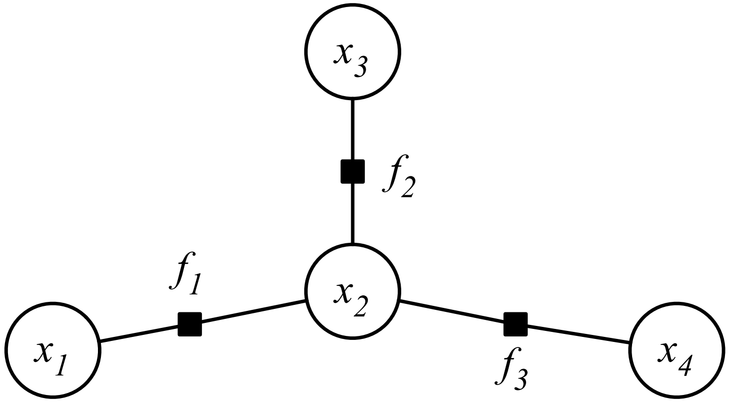
$$\mu_{f \to x}(x) = \sum_{\mathbf{x} \backslash x} \left(f(\mathbf{x}) \prod_{y \in \mathrm{n}(f) \backslash \{x\}} \mu_{y \to f}(y) \right)$$

If a variable has only one factor as a neighbor, it can initiate message propagation.

Once a variable has received all messages from its neighboring function nodes we can compute the probability of that variable by multiplying all the messages and renormalising:

$$p(x) \propto \prod_{h \in \mathrm{n}(x)} \mu_{h \to x}(x)$$

Propagation in Factor Graphs

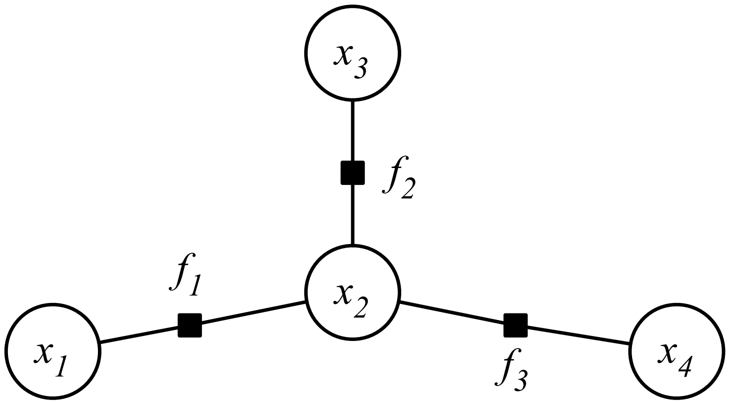


initialise all messages to be 1

an example schedule of messages resulting in computing $p(x_4)$:

message direction	message value
$x_1 \to f_1$	$1(x_1)$
$x_3 \to f_2$	$1(x_3)$
$f_1 \to x_2$	$\sum_{x_1} f_1(x_1, x_2) 1(x_1)$
$f_2 \to x_2$	$\sum_{x_3} f_2(x_3, x_2) 1(x_3)$
$x_2 \to f_3$	$\left(\sum_{x_1} f_1(x_1, x_2)\right)\left(\sum_{x_3} f_2(x_3, x_2)\right)$
$f_3 \to x_4$	$\sum_{x_2} f_3(x_2, x_4)\left(\sum_{x_1} f_1(x_1, x_2)\right)\left(\sum_{x_3} f_2(x_3, x_2)\right)$

Propagation in Factor Graphs



initialise all messages to be 1

an example schedule of messages resulting in computing $p(x_4|x_1 = a)$:

message direction	message value
$x_1 \rightarrow f_1$	$\delta(x_1 = a)$
$x_3 \rightarrow f_2$	$1(x_3)$
$f_1 \rightarrow x_2$	$\sum_{x_1} f_1(x_1, x_2)\delta(x_1 = a) \; = \; f_1(x_1 = a, x_2)$
$f_2 \rightarrow x_2$	$\sum_{x_3} f_2(x_3, x_2)1(x_3)$
$x_2 \rightarrow f_3$	$f_1(x_1 = a, x_2) \left(\sum_{x_3} f_2(x_3, x_2)\right)$
$f_3 \rightarrow x_4$	$\sum_{x_2} f_3(x_2, x_4) f_1(x_1 = a, x_2) \left(\sum_{x_3} f_2(x_3, x_2)\right)$

Elimination Rules for Factor Graphs

- **eliminating observed variables**

  If a variable x_i is **observed**, i.e. its value is given, then it is a *constant* in all functions that include x_i.

  We can **eliminate** x_i from the graph by removing the corresponding node and modifying all neighboring functions to treat it as a constant.

Elimination Rules for Factor Graphs

- **eliminating hidden variables**

  If a variable x_i is **hidden** and we are not interested in it we can eliminate it from the graph by summing over all its values.

$$
\begin{aligned}
\sum_{x_i} p(\mathbf{x}) &= \frac{1}{Z} \sum_{x_i} \prod_j f_j(\mathbf{x}_{S_j}) \\
&= \frac{1}{Z} \prod_{j \notin \mathrm{n}(x_i)} f_j(\mathbf{x}_{S_j}) \left(\sum_{x_i} \prod_{k \in \mathrm{n}(x_i)} f_k(\mathbf{x}_{S_k}) \right) \\
&= \frac{1}{Z} \prod_{j \notin \mathrm{n}(x_i)} f_j(\mathbf{x}_{S_j}) \quad f_{\mathrm{new}}(\mathbf{x}_{S_{\mathrm{new}}})
\end{aligned}
$$

where $f_{\mathrm{new}}(\mathbf{x}_{S_{\mathrm{new}}}) = \sum_{x_i} \prod_{k \in \mathrm{n}(x_i)} f_k(\mathbf{x}_{S_k})$ and $S_{\mathrm{new}} = \bigcup\limits_{k \in \mathrm{n}(x_i)} S_k \setminus \{i\}$.

This causes all its neighboring function nodes to merge into one new function node.