Mauna Loa Atmospheric Carbon Dioxide Model

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This short document explains some technical details on the Gaussian Process (GP) model used to make the figure `https://mlg.eng.cam.ac.uk/carl/words/keeling.pdf` discussed here `https://mlg.eng.cam.ac.uk/carl/words/carbon.html`, showing the concentration and growth rate of carbon dioxide measured at Mauna Loa. The motivation for providing the figure is to help interpret the data in a statistically sound way. The figure itself is not included here, as it's updated regularly on the web as more measurements become available, but the methodology is fixed.

1 The Observations

The data can be found at `https://gml.noaa.gov/webdata/ccgg/trends/co2/co2_mm_mlo.txt`, and consists of monthly average values of atmospheric carbon dioxide concentrations, starting in 1958. Thank you to the National Oceanic and Atmospheric Administration (NOAA) for providing the data. I only use columns 3 and 4, named `decimal date` and `monthly average`, I will refer to these as time, t, [years] and carbon dioxide concentration, c, [ppm] (parts per million). The objective is to build a model of the concentration as a function of time $c(t)$.

2 The Gaussian Processes model

The model is based on a prior Gaussian process:

$$c \sim \mathcal{N}(\mu, k_{\text{trend}} + k_{\text{season}} + k_{\text{noise}}), \tag{1}$$

which means: the function c is distributed according to a Gaussian process with mean function μ and covariance function which is a sum of three components: the trend k_{trend}, the seasonal part k_{season} and the noise k_{noise}. The posterior and marginal likelihood are computed by conditioning on the observations, and the marginal likelihood is optmized wrt the parameters (sometimes called hyperparameters) of the covariance function.

2.1 The mean

The prior mean function is the constant $\mu(t) = 280$ ppm, the pre-industrial carbon dioxide concentration.

2.2 The trend

The prior trend is a sum of three types of random walk:

$$k_{\mathrm{trend}}(t,t') \;=\; \sigma_B^2 k_{\mathrm{Brown}}(t,t') + \sigma_2^2 k_{2\mathrm{IW}}(t,t') + \sigma_3^2 k_{3\mathrm{IW}}(t,t'), \tag{2}$$

where k_{Brown} is the covariance of Brownian motion (equivalent to integrated white noise), $k_{2\mathrm{IW}}$ is the covariance of twice integrated white noise and $k_{3\mathrm{IW}}$ is the covariance of 3 times integrated white noise. Brownian motion is rough, twice integrated white noise is smoother and 3 times integrated white noise is smoother still; the parameters σ_B, σ_2 and σ_3 control their relative magnitudes. In detail

$$\begin{aligned} k_{\mathrm{Brown}}(t,t') &= \min(t,t'), \\ k_{2\mathrm{IW}}(t,t') &= \tfrac{1}{3}\min^3(t,t') + \tfrac{1}{2}|t-t'|\min^2(t,t'), \\ k_{3\mathrm{IW}}(t,t') &= \tfrac{1}{20}\min^5(t,t') + \tfrac{1}{8}|t-t'|\min^4(t,t') + \tfrac{1}{12}(t-t')^2\min^3(t,t'). \end{aligned} \tag{3}$$

Each random walk is anchored at zero in the year $t = 1750$, ie at the end of pre-industrial times. The optimized parameters are

parameter	meaning	value
σ_B	Brownian motion magnitude	0.51 ppm year$^{-1/2}$
σ_2	twice integrated white noise magnitude	0.0 ppm year$^{-3/2}$
σ_3	thrice integrated white noise magnitude	0.0014 ppm year$^{-5/2}$

It appears that the model shuts down the twice integrated noise contribution as unnecessary.

2.3 Seasonal contribution

The prior seasonal covariance is

$$k_{\mathrm{season}}(t,t') \;=\; \sigma_s^2 k_{\mathrm{periodic}}(t,t')\, k_{\mathrm{Gauss}}(t,t') \tag{4}$$

which is a distribution over smooth locally periodic functions with unit period, where the extent of "local" is determined by the length scale of k_{Gauss}. In detail:

$$k_{\mathrm{periodic}}(t,t') \;=\; \frac{\exp(-\frac{2\sin^2(\pi(t-t'))}{\ell^2}) - \alpha}{1-\alpha}, \quad \alpha \;=\; \int_0^{\pi} \exp(-\frac{2\sin^2(r)}{\ell^2})dr. \tag{5}$$

The shape of the periodic function is controlled by ℓ, very large values of ℓ will cause the functions to be close to pure sine functions, and small values giving more elaborate smooth periodic functions. Finally:

$$k_{\mathrm{Gauss}}(t,t') \;=\; \exp(-\frac{(t-t')^2}{2\lambda^2}), \tag{6}$$

with characteristic length scale λ. The optimized parameters are

parameter	meaning	value
σ_s	seasonal magnitude	1.52 ppm
ℓ	seasonal shape	1.34
λ	seasonal decay time	141 years

The very long seasonal decay time shows that he shape of the seasonal component changes only slowly.

2.4 Noise

The prior noise covariance contribution is

$$\begin{aligned} k_{\text{noise}}(t, t') &= \sigma_n^2 k_{\text{white}}(t, t') + \sigma_c^2 k_{\text{Gauss}}(t, t') \\ &= \sigma_n^2 \delta(t = t') + \sigma_c^2 \exp(-\frac{(t - t')^2}{2\ell_n^2}), \end{aligned} \tag{7}$$

where $\delta(t = t')$ is Kronecker's delta, one if and only if $t = t'$. The noise is the sum of uncorrelated white noise and noise with correlation length scale ℓ_n. The optimized parameters are

parameter	meaning	value
σ_n	white noise magnitude	0.19 ppm
σ_c	correlated noise magnitude	0.14 ppm
ℓ_n	correlation length	5.1 weeks

The combined inferred noise std deviation is $\sqrt{\sigma_n^2 + \sigma_c^2} = 0.23$ ppm, which agrees well with the uncertainties specified at the data source (but the model doesn't use these).

3 Carbon dioxide growth rates

The derivative of the posterior process does not exist, because Brownian motion isn't differentiable. For the growth rates, the distribution of finite differences (divided by the time distance) are plotted for different time distances.