Document models

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November 15th, 2017

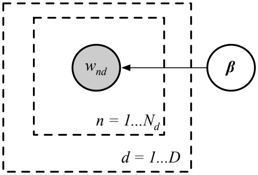
Key concepts

- a simple document model
- a mixture model for document
- fitting the mixture model with EM

A really simple document model

Consider a collection of D documents from a vocabulary of M words.

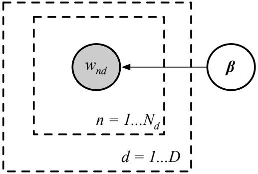
- N_d: number of words in document d.
- w_{nd}: n-th word in document d ($w_{nd} \in \{1 \ldots M\}$).
- $w_{nd} \sim \text{Cat}(\boldsymbol{\beta})$: each word is drawn from a discrete categorical distribution with parameters $\boldsymbol{\beta}$
- $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_M]^\top$: parameters of a categorical / multinomial distribution[1] over the M vocabulary words.



[1] It's a categorical distribution if we observe the sequence of words in the document, it's a multinomial if we only observe the counts.

A really simple document model

Modelling D documents from a vocabulary of M unique words.



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- $w_{nd} \sim \text{Cat}(\boldsymbol{\beta})$: each word is drawn from a discrete categorical distribution with parameters $\boldsymbol{\beta}$

We can fit $\boldsymbol{\beta}$ by maximising the likelihood:

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= \text{argmax}_{\boldsymbol{\beta}} \prod_{d=1}^{D} \prod_{n}^{N_d} \text{Cat}(w_{nd}|\boldsymbol{\beta}) \\ &= \text{argmax}_{\boldsymbol{\beta}} \text{Mult}(c_1, \ldots, c_M|\boldsymbol{\beta}, N)\end{aligned}$$

$$\boxed{\hat{\beta}_m = \frac{c_m}{N} = \frac{c_m}{\sum_{\ell=1}^{M} c_\ell}}$$

- $N = \sum_{d=1}^{D} N_d$: total number of words in the collection.
- $c_m = \sum_{d=1}^{D} \sum_{n}^{N_d} \mathbb{I}(w_{nd} = m)$: total count of vocabulary word m.

Maximum Likelihood and Lagrange multipliers

In maximum likelihood learning, we want to maximize the (log) likelihood

$$p(\mathbf{w}|\boldsymbol{\beta}) \;=\; \prod_{n=1}^{D}\prod_{n=1}^{N_d} \beta_{w_{nd}} \;=\; \prod_{m=1}^{M} \beta_m^{c_m}, \;\; \text{or} \;\; \log p(\mathbf{w}|\boldsymbol{\beta}) \;=\; \sum_{m=1}^{M} c_m \log \beta_m,$$

subject to the normalizing constraint that $\sum_{m=1}^{M} \beta_m = 1$.
An easy way to do this optimization is to add the Lagrange multiplier to the cost

$$\mathsf{F} \;=\; \sum_{m=1}^{M} c_m \log \beta_m + \lambda(1 - \sum_{m=1}^{M} \beta_m),$$

taking derivatives and setting to zero, we obtain

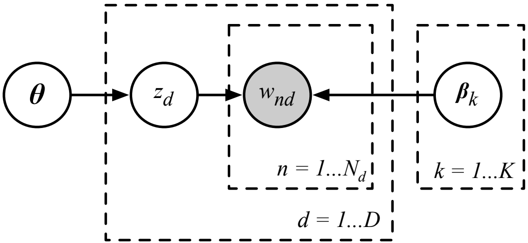
$$\frac{\partial \mathsf{F}}{\partial \beta_m} \;=\; \frac{c_m}{\beta_m} - \lambda \;=\; 0 \;\Rightarrow\; \beta_m \;=\; \frac{c_m}{\lambda} \;\text{ and }\; \frac{\partial \mathsf{F}}{\partial \lambda} \;=\; 0 \;\Rightarrow\; \sum_{m=1}^{M} \beta_m \;=\; 1,$$

which we combine to $\beta_m = c_m/n$, where n is the total number of words.

Limitations of the really simple document model

- Document d is the result of sampling N_d words from the categorical distribution with parameters $\boldsymbol{\beta}$.
- $\boldsymbol{\beta}$ estimated by maximum likelihood reflects the aggregation of all documents.
- All documents are therefore modelled by the global word frequency distribution.
- This generative model does not specialise.
- We would like a model where different documents might be about different *topics*.

A mixture of categoricals model



$$z_d \sim \text{Cat}(\theta)$$
$$w_{nd}|z_d \sim \text{Cat}(\beta_{z_d})$$

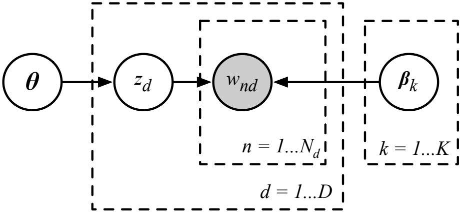
We want to allow for a mixture of K categoricals parametrised by $\beta_1, \ldots, \beta_K$. Each of those categorical distributions corresponds to a *document category*.

- $z_d \in \{1, \ldots, K\}$ assigns document d to one of the K categories.
- $\theta_k = p(z_d = k)$ is the probability any document d is assigned to category k.
- so $\theta = [\theta_1, \ldots, \theta_K]$ is the parameter of a categorical distribution over K categories.

We have introduced a new set of *hidden* variables z_d.

- How do we fit those variables? What do we do with them?
- Are these variables interesting? Or are we only interested in θ and β?

A mixture of categoricals model: the likelihood



$$
\begin{aligned}
z_d &\sim \text{Cat}(\theta) \\
w_{nd}|z_d &\sim \text{Cat}(\boldsymbol{\beta}_{z_d})
\end{aligned}
$$

$$
\begin{aligned}
p(\mathbf{w}|\theta, \boldsymbol{\beta}) &= \prod_{d=1}^{D} p(\mathbf{w}_d|\theta, \boldsymbol{\beta}) \\
&= \prod_{d=1}^{D} \sum_{k=1}^{K} p(\mathbf{w}_d, z_d = k|\theta, \boldsymbol{\beta}) \\
&= \prod_{d=1}^{D} \sum_{k=1}^{K} p(z_d = k|\theta) p(\mathbf{w}_d|z_d = k, \boldsymbol{\beta}_k) \\
&= \prod_{d=1}^{D} \sum_{k=1}^{K} p(z_d = k|\theta) \prod_{n=1}^{N_d} p(w_{nd}|z_d = k, \boldsymbol{\beta}_k)
\end{aligned}
$$

EM and Mixtures of Categoricals

In the mixture model, the likelihood is:

$$p(\mathbf{w}|\theta, \boldsymbol{\beta}) = \prod_{d=1}^{D} \sum_{k=1}^{K} p(z_d = k|\theta) \prod_{n=1}^{N_d} p(w_{nd}|z_d = k, \boldsymbol{\beta}_k)$$

E-step: for each d, set q to the posterior (where $c_{md} = \sum_{n=1}^{N_d} \mathbb{I}(w_{nd} = m)$):

$$q(z_d = k) \;\propto\; p(z_d = k|\theta) \prod_{n=1}^{N_d} p(w_{nd}|\beta_{k,w_n}) \;=\; \theta_k \operatorname{Mult}(c_{1d}, \ldots, c_{Md}|\boldsymbol{\beta}_k, N_d) \;\stackrel{\text{def}}{=}\; r_{kd}$$

M-step: Maximize

$$\begin{aligned}
\sum_{d=1}^{D}\sum_{k=1}^{K} q(z_d = k) \log p(\mathbf{w}, z_d) &= \sum_{k,d} r_{kd} \log \Big[p(z_d = k|\theta) \prod_{n=1}^{N_d} p(w_{nd}|\beta_{k,w_{nd}}) \Big] \\
&= \sum_{k,d} r_{kd} \Big(\log \prod_{m=1}^{M} \beta_{km}^{c_{md}} + \log \theta_k \Big) \\
&= \sum_{k,d} r_{kd} \Big(\sum_{m=1}^{M} c_{md} \log \beta_{km} + \log \theta_k \Big) \;\stackrel{\text{def}}{=}\; F(R, \theta, \beta)
\end{aligned}$$

EM: M step for mixture model

$$F(R, \theta, \beta) = \sum_{k,d} r_{kd} \Big(\sum_{m=1}^{M} c_{md} \log \beta_{km} + \log \theta_k \Big)$$

Need Lagrange multipliers to constrain the maximization of F and ensure proper distributions.

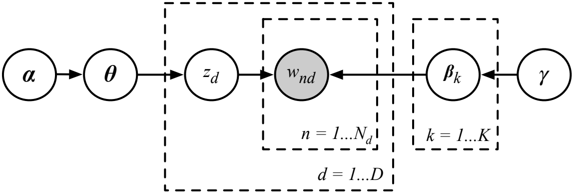
$$\hat{\theta}_k \leftarrow \text{argmax}_{\theta_k} \;\; F(R, \theta, \beta) + \lambda\Big(1 - \sum_{k'=1}^{K} \theta_{k'}\Big)$$

$$= \frac{\sum_{d=1}^{D} r_{kd}}{\sum_{k'=1}^{K} \sum_{d=1}^{D} r_{k'd}} = \frac{\sum_{d=1}^{D} r_{kd}}{D}$$

$$\hat{\beta}_{km} \leftarrow \text{argmax}_{\beta_{km}} \;\; F(R, \theta, \beta) + \sum_{k'=1}^{K} \lambda_{k'}\Big(1 - \sum_{m'=1}^{M} \beta_{k'm'}\Big)$$

$$= \frac{\sum_{d=1}^{D} r_{kd} c_{md}}{\sum_{m'=1}^{M} \sum_{d=1}^{D} r_{kd} c_{m'd}}$$

A Bayesian mixture of categoricals model



$$
\begin{aligned}
\boldsymbol{\theta} &\sim \mathrm{Dir}(\alpha) \\
\boldsymbol{\beta}_k &\sim \mathrm{Dir}(\gamma) \\
z_d|\boldsymbol{\theta} &\sim \mathrm{Cat}(\boldsymbol{\theta}) \\
w_{nd}|z_d, \boldsymbol{\beta} &\sim \mathrm{Cat}(\boldsymbol{\beta}_{z_d})
\end{aligned}
$$

With the EM algorithm we have essentially estimated $\boldsymbol{\theta}$ and $\boldsymbol{\beta}$ by maximum likelihood. An alternative, Bayesian treatment infers these parameters starting from priors, e.g.:

- $\boldsymbol{\theta} \sim \mathrm{Dir}(\alpha)$ is a symmetric Dirichlet over category probabilities.
- $\boldsymbol{\beta}_k \sim \mathrm{Dir}(\gamma)$ are symmetric Dirichlets over vocabulary probabilities.

What is different?

- We no longer want to compute a point estimate of $\boldsymbol{\theta}$ or $\boldsymbol{\beta}$.
- We are now interested in computing the *posterior* distributions.