Linear in the parameters models and GP

Hong Ge

Oct, 2025

Key concepts

- Linear in the parameters model correspond to Gaussian processes
- explicitly calculate the GP from the linear model
  - mean function
  - covaraince function
- going from covariance function to linear model
  - done using Mercer's theorem
  - may not always result in a finite linear model
- computational consideration: which is best?

From random functions to covariance functions

Consider the class of linear functions:

$$f(x) \;=\; ax + b, \;\text{ where }\; a \sim \mathcal{N}(0, \alpha), \;\text{ and }\; b \sim \mathcal{N}(0, \beta).$$

We can compute the mean function:

$$\mu(x) \;=\; E[f(x)] \;=\; \iint f(x)p(a)p(b)\,da\,db \;=\; \int axp(a)\,da + \int bp(b)\,db = 0,$$

and covariance function:

$$\begin{aligned} k(x, x') \;&= E[(f(x) - 0)(f(x') - 0)] \;=\; \iint (ax + b)(ax' + b)p(a)p(b)\,da\,db \\ &= \int a^2 xx' p(a)\,da + \int b^2 p(b)\,db + (x + x') \int ap(a)p(b)\,da\,db = \alpha xx' + \beta. \end{aligned}$$

Therefore: a linear model with Gaussian random parameters corresponds to a GP with covariance function $k(x, x') = \alpha xx' + \beta$.

From finite linear models to Gaussian processes (1)

Finite linear model with Gaussian priors on the weights:

$$f(x) = \sum_{m=1}^{M} w_m \, \phi_m(x) \qquad\qquad p(\boldsymbol{w}) = \mathcal{N}(\boldsymbol{w};\; 0, A)$$

The joint distribution of any $\mathbf{f} = [f(x_1), \ldots, f(x_N)]^\top$ is a multivariate Gaussian – this looks like a Gaussian Process!

The prior $p(\mathbf{f})$ is fully characterized by the *mean* and *covariance* functions.

$$m(x) = E_{\boldsymbol{w}}\big(f(x)\big) = \int \Big(\sum_{m=1}^{M} w_k \phi_m(x) \Big) p(\boldsymbol{w}) d\boldsymbol{w} = \sum_{m=1}^{M} \phi_m(x) \int w_m p(\boldsymbol{w}) d\boldsymbol{w}$$

$$= \sum_{m=1}^{M} \phi_m(x) \int w_m p(w_m) dw_m = 0$$

The *mean function* is zero.

From finite linear models to Gaussian processes (2)

Covariance function of a finite linear model

$$f(x) = \sum_{m=1}^{M} w_m \phi_m(x) = \boldsymbol{w}^\top \boldsymbol{\phi}(x)$$
$$p(\boldsymbol{w}) = \mathcal{N}(\boldsymbol{w};\ 0, A)$$

$$\boldsymbol{\phi}(x) = [\phi_1(x), \ldots, \phi_M(x)]^\top{}_{(M\times 1)}$$

$$k(x_i, x_j) = \mathrm{Cov}_{\boldsymbol{w}\left(f(x_i), f(x_j)\right)} = \mathrm{E}_{\boldsymbol{w}\left(f(x_i) f(x_j)\right)} - \underbrace{\mathrm{E}_{\boldsymbol{w}\left(f(x_i)\right)} \mathrm{E}_{\boldsymbol{w}\left(f(x_j)\right)}}_{0}$$

$$= \int \ldots \int \Big(\sum_{k=1}^{M} \sum_{l=1}^{M} w_k w_l \phi_k(x_i) \phi_l(x_j) \Big) p(\boldsymbol{w})\, d\boldsymbol{w}$$

$$= \sum_{k=1}^{M} \sum_{l=1}^{M} \phi_k(x_i)\phi_l(x_j) \underbrace{\iint w_k w_l p(w_k, w_l) dw_k dw_l}_{A_{kl}} = \sum_{k=1}^{M} \sum_{l=1}^{M} A_{kl} \phi_k(x_i) \phi_l(x_j)$$

$$\boxed{k(x_i, x_j) = \boldsymbol{\phi}(x_i)^\top A \boldsymbol{\phi}(x_j)}$$

Note: If $A = \sigma_{\boldsymbol{w}^2 \mathbf{I}}$ then $k(x_i, x_j) = \sigma_{\boldsymbol{w}^2 \sum_{k=1}^{M} \phi_k(x_i) \phi_k(x_j) = \sigma_{\boldsymbol{w}^2 \boldsymbol{\phi}(x_i)^\top \boldsymbol{\phi}(x_j)}}$

GPs and Linear in the parameters models are equivalent

We've seen that a Linear in the parameters model, with a Gaussian prior on the weights is also a GP.

Might it also be the case that every GP corresponds to a Linear in the parameters model?

The answer is yes, but not necessarily a finite one. (Mercer's theorem.)

Note the different computational complexity: GP: $\mathcal{O}(N^3)$, linear model $\mathcal{O}(NM^2)$ where M is the number of basis functions and N the number of training cases.

So, which representation is most efficient?