Discrete Categorical Distribution

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Adapted from Carl Edward Rasmussen

Key concepts

We generalize the concepts from binary variables to multiple discrete outcomes.

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- discrete and multinomial distributions
- the Dirichlet distribution

The multinomial distribution (1)

- Generalisation of the binomial distribution from 2 outcomes to m outcomes.
- Useful for random variables that take one of a finite set of possible outcomes.

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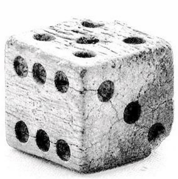

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Throw a die $n = 60$ times, and count the observed (6 possible) outcomes.

Outcome	Count
$X = x_1 = 1$	$k_1 = 12$
$X = x_2 = 2$	$k_2 = 7$
$X = x_3 = 3$	$k_3 = 11$
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Note: We have one parameter too many.
We don't need to know all the k_i and n, because $\sum_{i=1}^{6} k_i = n$.

The multinomial distribution (2)

Consider a discrete random variable X that can take one of m values $x_1, \ldots, x_m$.

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The probability of observing a vector of occurrences $\mathbf{k} = [k_1, \dots, k_m]^\top$ is given by the *multinomial distribution* parametrised by $\boldsymbol{\pi} = [\pi_1, \dots, \pi_m]^\top$:

$$p(\mathbf{k}|\boldsymbol{\pi}, n) \;=\; p(k_1, \dots, k_m|\pi_1, \dots, \pi_m, n) \;=\; \frac{n!}{k_1!k_2!\dots k_m!} \prod_{i=1}^{m} \pi_i^{k_i}$$

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The discrete or *categorical distribution* is the generalisation of the Bernoulli to m outcomes, and the special case of the multinomial with one trial:

$$p(X = x_i|\boldsymbol{\pi}) = \pi_i.$$

Example: word counts in text

Consider describing a text document by the frequency of occurrence of every distinct word.

The UCI *Bag of Words* dataset from the University of California, Irvine. [1]

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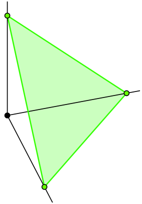
- Vector $\boldsymbol{\pi}$ lives in the open standard $m-1$ simplex.
- $\boldsymbol{\pi}$ could for example be the parameter vector of a multinomial. [Figure on the right $m = 3$.]



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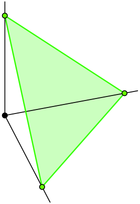

The Dirichlet distribution is given by

$$\mathrm{Dir}(\boldsymbol{\pi}|\alpha_1, \ldots, \alpha_m) \;=\; \frac{\Gamma(\sum_{i=1}^m \alpha_i)}{\prod_{i=1}^m \Gamma(\alpha_i)} \prod_{i=1}^m \pi_i^{\alpha_i - 1} \;=\; \frac{1}{B(\boldsymbol{\alpha})} \prod_{i=1}^m \pi_i^{\alpha_i - 1}$$

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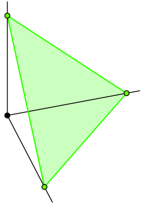

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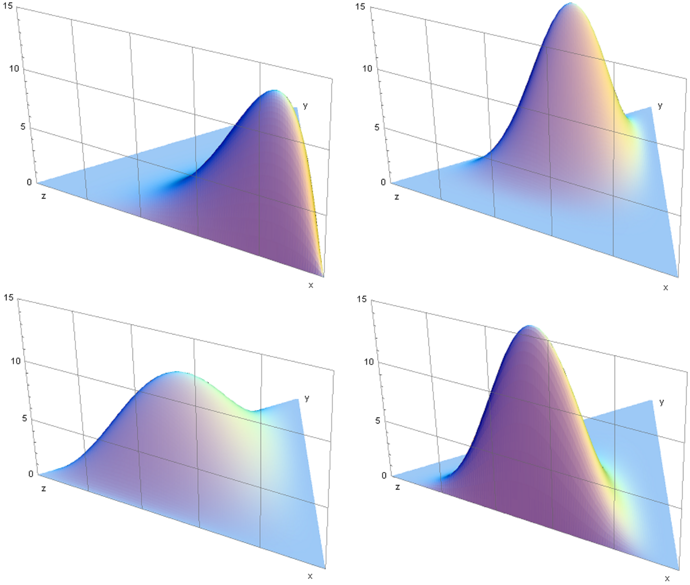
- $\boldsymbol{\alpha} = [\alpha_1, \dots, \alpha_m]^\top$ are the shape parameters.
- $B(\boldsymbol{\alpha})$ is the multivariate beta function.
- $E(\pi_j) = \frac{\alpha_j}{\sum_{i=1}^m \alpha_i}$ is the mean for the j-th element.

Dirichlet Distributions from Wikipedia



The symmetric Dirichlet distribution

In the symmetric Dirichlet distribution all parameters are identical: $\alpha_i = \alpha, \ \ \forall i$.

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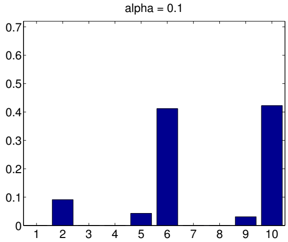

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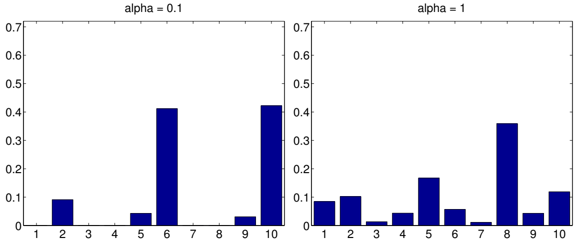

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- Middle: $\alpha = 1.0$ (Uniform)

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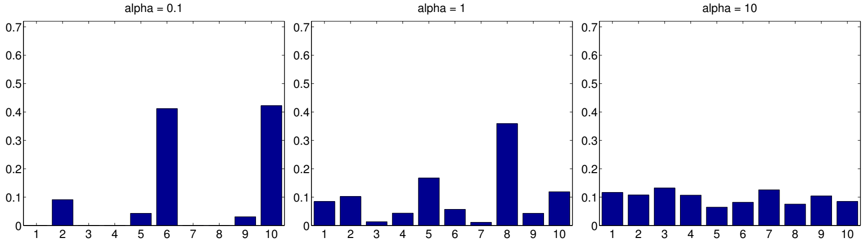

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- Middle: $\alpha = 1.0$ (Uniform)
- Right: $\alpha = 10$ (Peaked at center)